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How Euler Did It

by Ed Sandifer

Euler as a Teacher – Part 2

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Let us start with the Great Quotation, dubiously attributed to Laplace by Guglielmo Libri[1] about 1846

Lisez Euler, lisez Euler, c'est notre maître à tous.

We traditionally translate this as

Read Euler, read Euler. He is the master of us all.

This gave Bill Dunham a title befitting his most excellent book, [Dunham 1998] but there are other ways to translate it. Because $\text{maître} = \begin{cases}\text{master}\\ \text{teacher}\end{cases}$ and "*notre … à tous*" can mean "of us all" or "*notre*" can be assigned to modify "*maître*", leaving "*à tous*" to mean "all things", other valid translations include:

Read Euler, read Euler. He is our master in all things.
Read Euler, read Euler. He is the teacher of us all.
Read Euler, read Euler. He is our teacher in all things.
etc.

St. Petersburg 1766-1783

In Part 1 of this column [Sandifer Jan 2010] we looked at what is known about Euler the Teacher during his first St. Petersburg period (1727-1741) and his time in Berlin (1741-1766). Condorcet [Condorcet 1786] gives us some accounts of Euler's teaching in his second St. Petersburg period:

His sons and students copied his calculations and wrote by dictation the remaining *Mémoires*.

[1] Libri was a scoundrel, a forger, a book thief and an indifferent mathematician, [Rice 2003] but he did write a decent history of mathematics. In Libri's defense, note that he claims that he heard these words "*de sa propre bouche*", from Laplace's own mouth, not that Laplace actually wrote them down. [WikiQuote]

It can be seen that he much preferred the education of his students than the small satisfaction derived from astonishment; he never believed that he had truly done enough for Science if he did not feel that that he had added new truths to enrich it and the exposure of the simplicity of the idea which lead him there.

…

Of the sixteen professors attached to the Saint Petersburg Academy eight were trained under him and all are known through their works and have been awarded various academic distinctions and are proud to add the title of Euler's disciples.

Condorcet goes on to mention some of the people who studied under Euler, his two sons, Lexell, and Fuss in particular, and that Fuss married one of Euler's granddaughters.

Except for disclaiming "the small satisfaction derived from astonishment," this does little to tell us about how Euler taught or why he was effective. The stories that Condorcet relates about:

- Euler's students taking dictation,
- Euler reading things written large on a tablet or chalkboard,
- Euler wearing a shiny track in the table as he used it to guide himself while he paced around it and talked to his students,

tell us very little about how Euler actually taught.

Euler last attended a meeting of the St. Petersburg Academy on January 16, 1777, after which he sent his papers in to the Academy with his assistants. One of the portraits of Euler, shown above, has a sub-portrait, a smaller rectangle beneath the oval of the main portrait. The sub-portrait shows two men, one with pen and paper, sitting at a table. Apparently it pictures Euler dictating to one of his assistants, probably his son, Johann Albrecht, because Euler himself could no longer read or write.

Internal evidence

I want to cite two kinds of evidence about Euler's teaching in St. Petersburg: 1° data from the *Adversariis mathematicis*, or Mathematical daybook [E806]; and 2° subjective observations from reading several of Euler's late papers.

The *Adversariis mathematicis*.

The *Adversariis mathematicis* was a series of three notebooks kept in the foyer of the St. Petersburg Academy. Members used the *Adversariis* as a kind of chat room or virtual seminar to show their colleagues what they were working on and to announce preliminary results. Eventually, the

Adversariis filled three notebooks totaling 776 pages. Less than 30% of their contents, amounting to 111 entries, appeared in the *Opera posthuma* in 1862. [E805] They are sprinkled about several volumes of the *Opera omnia* according to the subjects of the notes.

Most of the notes are dull and technical. Many are wrong. Some are dead ends, sometimes intriguing, but ultimately doomed. For example, Note 67 is a 17-page joint effort by J. A. Euler, Lexell, Fuss and Krafft to solve Fermat's Last Theorem. They get stuck on the same technical points of unique factorization that befuddled 19th century mathematicians.

Note 24 is a contribution by Krafft, noting that both $x^2 + x + 17$ and $x^2 + x + 41$ give nothing but prime numbers for small values of x.

Note 104 is by J. A. Euler, and essentially he rewrites Ptolmey's theorem about the sides and diagonals of a cyclic quadrilateral in terms of sines, and then extends it to "infinitely large" circles, i. e. straight lines. It is not clear if he noticed that what he gets is a theorem in geometric algebra found in Euclid's *Elements*, book II.

Note 96 is a proof by Nicolas Fuss of the elementary properties of the Euler φ function. Notes 82 and 83 are about magic squares and Greco-Latin squares. Dozens are about Diophantine equations, especially those related to quadratic reciprocity.

The full text of the *Adversariis mathematicis* has not been published, and I've not seen any description of what fills the more than 70% of the notebooks that were not published in 1862. We can only speculate.

By the time these notes were published in 1862, 79 years after Euler's death, the results were all quite stale, but only a couple would have been interesting even if they had been disseminated earlier.

I tabulated who contributed to the *Adversariis mathematiciis*:

	number	pages
Fuss	26	68
Lexell	23	70
JA Euler	17	51
Krafft	11	42
Golovin	3	6
Euler	1	2
unsigned	56	

Note that the number of signatures doesn't add up to 55, because several notes, like Note 67, were joint efforts with more than one signature. Half the notes were unsigned.

What does this say about Euler's teaching? It looks to me like Euler used the *Adversariis* as a proxy for a graduate seminar. It is as if his five students would read papers from earlier in Euler's career, work through the difficulties, either alone or together, and demonstrate their mastery of the material (or ineptitude) by proving old results in new ways, working through examples, filling in details and extending and generalizing the results. The style closely resembles the way Johann Bernoulli had taught Euler some sixty years earlier.

Late papers

When Euler died in 1783, he left a legacy of over 200 unpublished papers, virtually all of which he wrote after returning to St. Petersburg in 1766. Only a few of them were important. Let's look at a couple of examples.

In *Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae*, "Investigation of certain series which are designed to approximate the true ratio of the circumference of a circle to its diameter very closely," [E705, Sandifer Feb 2009] Euler repeats work from *De variis modis circuli quadraturam numeris proxime exprimendi*," On several means of expressing the quadrature of area of a circle very accurately." [E74] The first paper showed how to use the Machin equations,

$$\arctan 1 = \arctan\frac{1}{a} + \arctan\frac{1}{b}$$

to give fast converging approximations to π. The second paper does the same thing, but engineers the series so they have easy denominators that are powers of 2, 5 and 10.

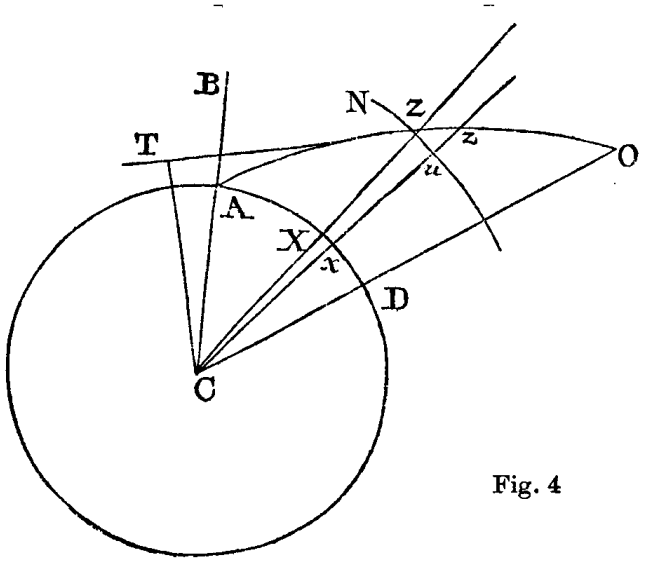

Fig. 4

Sur l'effet de la réfraction dans les observations terrestres, "On the effect of refraction on terrestrial observations," [E502] shows how to correct for the way the varying density of the atmosphere bends light and affects surveying the heights of mountains. It repeats much of the material from *De la réfraction de la lumière en passant par l'atmosphère selon les divers degrés tant de la chaleur que de l'élasticité de l'air*, "On the refraction of light passing through the atmosphere due to the different degrees of heat as well as the elasticity of the air," [E219] where he studied the same refraction for astronomical observations.

Many of his other late papers are like this, especially his late number theory, revisiting earlier problems with a new twist or complication, a different proof (not necessarily better) or a longer example. The papers are often a bit choppy and the examples less interesting than in his earlier papers. For a long time, I thought that these were signs of Euler declining as he aged. That may be true, but now I think that these are also the fingerprints of Euler's students. It seems likely that Euler would revisit an old paper and ask his students to see what they could do with it. They could have asked him questions, harvested ideas for extensions, and then, with his guidance, write papers under Euler's name. This would not be much different from a master artist who had his students fill in the backgrounds of his paintings.

Punch line

Euler learned at the feet of Johann Bernoulli, who had Euler "read the masters." Euler read difficult mathematics and Bernoulli helped him when he got stuck.

We now know some details of Euler's classroom teaching during his first St. Petersburg years, but we have no evidence or testimonies about what kind of teacher he was.

We only know a few details about Euler's teaching in Berlin. The setting wasn't a formal, but it foreshadowed his later teaching style.

In his late years, I propose that Euler was able to teach in the style under which he himself had learned. He had learned guided by the principle

Read the masters.

He taught under the style

Read me, read me. I am Euler and I am your teacher in all things.

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[Sandifer Jan 2010] Sandifer, C. Edward, *How Euler Did It*, No. 74, maa.org.

[WikiQuote] http://en.wikiquote.org/wiki/Pierre-Simon_Laplace

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