

	<h1 style="margin: 0;">How Euler Did It</h1> <p style="font-size: 1.5em; margin: 0;">by Ed Sandifer</p>	
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## Bending light

August 2009

In our last column we began a study of one of Euler's papers on how the Earth's atmosphere refracts light, *Sur l'effet de la réfraction dans les observations terrestres*, "On the effect of refraction on terrestrial observations", [E502] written about 1777 and published in 1780. We learned "that the rays of light do not always go in straight lines to our eyes, as we ordinarily suppose, but they are found to be a little bit curved, and their concavity is turned downward" and that this phenomenon is due in part to refraction as the rays pass between the rarified air at higher altitudes and the denser air at lower altitudes.

As we saw last month, in Euler's Fig. 3<sup>1</sup>,  $C$  represents the center of the earth and  $A$  the location of an observer. The circle is not the surface of the earth but all the points at the level of the observer. The point  $Q$  is some point above the observer and  $x$  is the vertical distance  $AQ$ . The density of the air at  $A$  and  $Q$  is denoted by  $c$  and  $q$  respectively (though  $q$  also denotes a point infinitely close to  $Q$ ). The air pressure, what Euler calls its elasticity, is denoted by  $k$  and  $p$  respectively. Because he used a mercury barometer to measure air pressure, he also used  $m$  to denote the relative density of mercury to air and told us that  $m$  was approximately 10,000. Then, by what we now call Boyle's law,

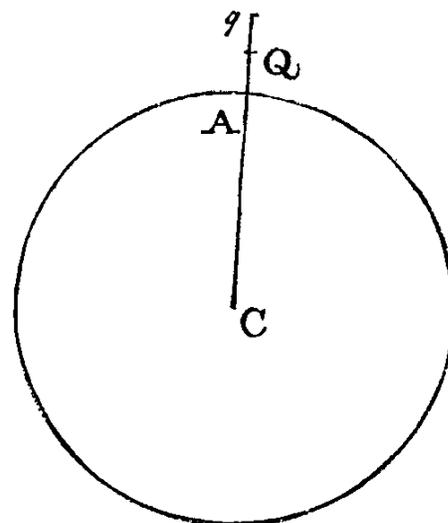


Fig. 3

$$\frac{q}{c} = \frac{p}{k}.$$

The main result of the first part of Euler's paper was an equation that describes the density of air at different altitudes. He found that

$$\frac{q}{c} = e^{-\frac{x}{mk}}.$$

<sup>1</sup> Our numbering of figures may be confusing here. We saw Euler's Figs. 1 and 3 in last month's column. We won't need to look at his Fig. 2. We've added an illustration between his Figs. 3 and 4 and named it Fig. 3.5.

He uses this formula in various forms, including its logarithmic forms,  $\frac{x}{mk} = \ln \frac{k}{p} = \ln \frac{c}{q}$ . He also notes that his formula for  $\frac{c}{q}$  is well approximated by the first few terms of a Taylor series expansion of its right-hand side, so for most practical purposes he would be able to use

$$\frac{c}{q} = 1 - \frac{x}{mk} + \frac{xx}{mmkk} - \frac{x^3}{m^3k^3}.$$

Indeed, because  $m$  is so large, "we could content ourselves for most observations to the first two terms,  $1 - \frac{x}{mk}$ , at least when we do not have to measure mountains of very considerable height."

Knowing the formulas for density, Euler is ready to move on to refraction. Euler asks us to consider a ray of light that passes directly from a vacuum into ordinary air of density  $c$ , which continues to denote the density of air at the point  $A$ , the level of the observer. Then there is a physical constant  $\delta$  such that the ratio of the sine of the incidence to the sine of the refraction is as 1 is to  $1 - \delta$ , and that the value of  $\delta$  is about  $\frac{3}{10,000}$ . In writing this, Euler is implicitly using Snell's law, illustrated in our Fig. 3.5.

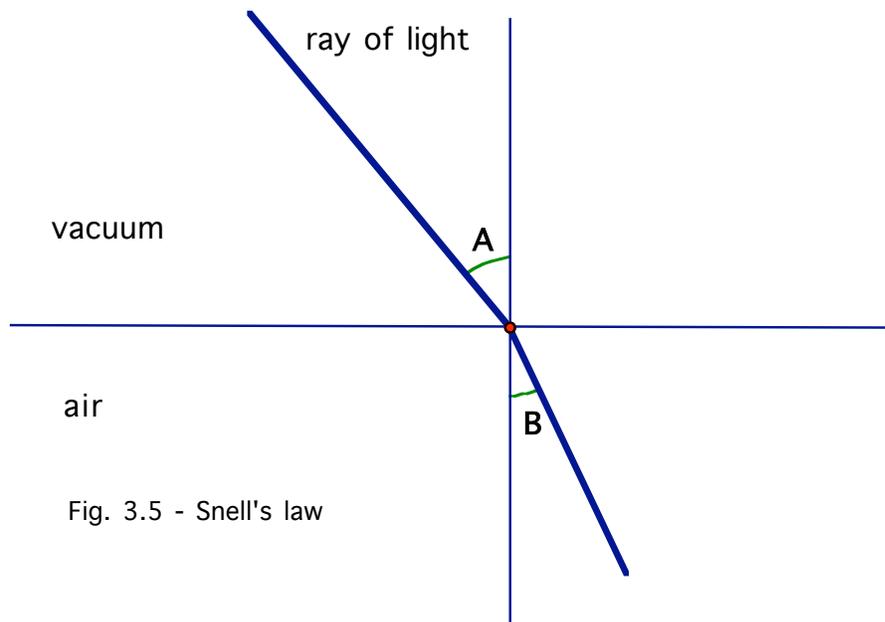


Fig. 3.5 - Snell's law

In Fig. 3.5, a ray of light is shown passing from a vacuum into air of density  $c$ . The angle  $A$  is the angle of incidence, or simply the incidence, and Euler calls the angle  $B$  the angle of refraction, or just the refraction. Then Euler's version of Snell's law says that  $\frac{\sin A}{\sin B} = \frac{1}{1 - \delta}$ . This version is formulated relative to the optical density of air at the observer. Modern versions use either the speed of light in the two media or they use tables of optical densities. They are all equivalent.

In Euler's problem, though, the density of air varies with altitude, so he has to adapt his formula a bit to describe light passing from a vacuum into air of density  $q$ . He claims that then the ratio of the sines will be "as 1 to  $1 - \frac{\delta q}{c}$ ", that is to say,  $\frac{\sin A}{\sin B} = \frac{1}{1 - \frac{\delta q}{c}}$

Likewise, if the same ray of light passes from a vacuum into air of density  $r$ , the ratio of the corresponding sines will be as 1 is to  $1 - \frac{\delta r}{c}$ . Moreover, the process reverses, so that for light passing the other direction, from air of density  $r$  into a vacuum, the corresponding ratio will be the inverse.

Euler claims that as a consequence, the ratio of the sines for a ray passing from air of density  $q$  to air of density  $r$  will be as  $1 - \frac{\delta q}{c}$  is to  $1 - \frac{\delta r}{c}$ . But because  $\delta$  is so very small, this is almost the same as the ratio 1 to  $1 + \frac{\delta(q-r)}{c}$ .

Now, using a standard technique of physics pioneered by Euler himself, Euler takes  $r$  to be infinitely close to  $q$ , which amounts to taking  $r = q + dq$ . Then as the ray passes from air of density  $q$  to air of density  $q + dq$ , the ratio of refraction will be as 1 is to  $1 - \frac{\delta dq}{c}$ . Reciprocally, as the light passes in the other direction, the ratio will be the reciprocal,  $1 - \frac{\delta dq}{c}$  to 1.

As we learned in last month's column, [Sandifer Jun 2009] the density of the atmosphere at an altitude  $x$  above the level of the observer at  $A$  is described by the equation  $\ln \frac{c}{q} = \frac{x}{mk}$ . Differentiating

this gives  $\frac{dq}{q} = -\frac{dx}{mk}$ , which can be rewritten

as  $dq = -\frac{q dx}{mk}$ . Substituting this into the last

form of his ratio of refraction makes that ratio as 1 is to  $1 - \frac{\delta q dx}{cmk}$ .

Now that he has formulas for the density of air and for the ratio of refraction, Euler is ready to set up and solve his main problem. He refers us to his rather complicated Fig. 4, where the circle represents the surface of the earth,  $A$ ,  $X$ ,  $x$  and  $D$  are points on the surface and  $C$  is the center. Then the radius of the earth is  $AC = a$ . Let  $O$  be an object directly above the point  $D$  that is seen by an observer at  $A$  by means of a ray of light that travels along the curve  $OzZA$ . He takes  $Z$  and  $z$  infinitely close together and assumes that they are directly above the points  $x$  and  $X$ , respectively.

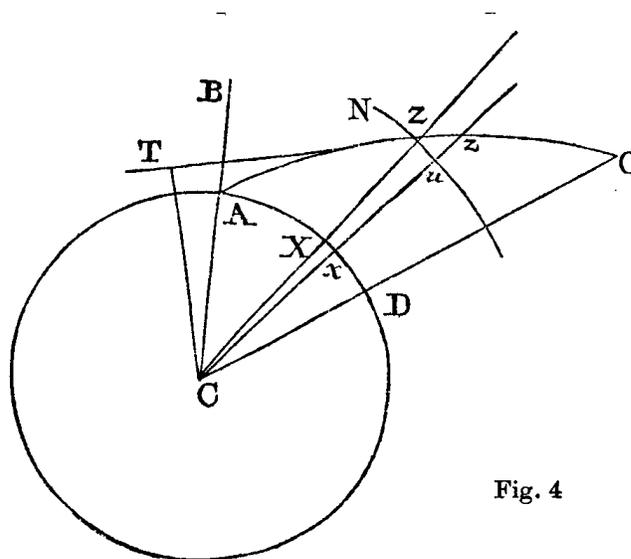


Fig. 4

Euler takes  $B$  to be a point directly above  $A$  for the sole purpose of being able to measure the angle  $BAZ = \zeta$ . He takes the angle  $ACZ = \phi$  and the distance  $CZ = z$ . Let  $x$  be the height of the point  $Z$  above  $X$  be  $x$  so that  $XZ = x$  and  $Z = a + x$  [taking  $Z$  to be both a distance and a point]. Thus for the point  $z$  we have the angle  $ACz = \phi + d\phi$ . This makes the angle  $ZCz = d\phi$  and the distance  $CZ = z + dz$ . Also

he draws the tangent  $ZT$  at the point  $Z$  and chooses the point  $T$  so that  $CT$  is perpendicular to the tangent.

Let  $CT = t$  and take the angle  $CZT = \omega$ . Then  $\sin \omega = \frac{t}{z}$ .

Now let  $NZu$  be the arc of a circle through  $Z$  with center  $C$ . It represents the layer of the atmosphere that passes through the point  $Z$ . Euler needs a new point  $s$ , not shown in Fig. 4, located beyond the point  $Z$  on the ray  $CZ$ , so that he can measure the angle of incidence of the ray  $OzZA$  as it passes through the layer  $NZu$ . Then angle  $zZs$  is the angle of incidence. We've just named this angle  $CZT = \omega$ , so angle  $CzZ = \omega + d\omega$ . Because angle  $zZs$  is an external angle of triangle  $CZz$ , it equals the sum of the other two angles  $CzZ$  and  $ZCz$ . Consequently, the angle of incidence is

$$zZs = \omega + d\omega + d\phi,$$

Because  $d\omega$  and  $d\phi$  are infinitesimals, Euler safely assumes that  $\cos(d\omega + d\phi) = 1$  and  $\sin(d\omega + d\phi) = d\omega + d\phi$ , so by the angle addition formula for the sine function, the sine of the angle of incidence is

$$\sin(\omega + d\omega + d\phi) = \sin \omega + (d\omega + d\phi)\cos \omega.$$

Meanwhile, the sine of the angle of refraction is  $\sin \omega$ . To apply Snell's law, we need the ratio of the sine of the incidence to the sine of the refraction. As fractions, this ratio is

$$\frac{\sin \omega + (d\omega + d\phi)\cos \omega}{\sin \omega} = 1 + \frac{(d\omega + d\phi)\cos \omega}{\sin \omega} = 1 + (d\omega + d\phi)\cot \omega.$$

Euler will find it more convenient if he can rewrite this fraction with a 1 in its numerator. Because the numerator of the fraction on the middle is an infinitesimal, he rewrites the ratio as

$$1 : 1 - \frac{(d\omega + d\phi)\cos \omega}{\sin \omega}.$$

When it comes time to use this ratio, though, he will use it in the form

$$1 : 1 - (d\omega + d\phi)\cot \omega.$$

That's one part of Snell's law. Now Euler must consider the effect of the density of air. Earlier he denoted the density of air at  $Z$  as  $q$ , so the density at  $z$  is  $q + dq$ . From his earlier work on refraction, we get that the ratio of the sines, what Euler calls the "ratio of refraction" is  $1 : 1 + \frac{\delta dq}{c}$ . Now, by Snell's law, the ratio of the sines equals the ratio of refraction, so we get

$$\frac{\delta dq}{c} = -(d\omega + d\phi)\cot \omega.$$

The first part of the article, the part we dealt with in last month's column, told us what we need to know to find  $dq$ . For the given height  $XZ = x$ , we have  $\ln \frac{c}{q} = \frac{x}{mk}$ , from which it follows that

$$dq = -\frac{q dx}{mk} = -\frac{q dz}{mk}.$$

Substituting this value for  $dq$  gives the equation

$$\frac{\delta q dz}{mck} = (d\omega + d\phi) \cot \omega.$$

Because  $\frac{q}{c} = e^{-\frac{x}{mk}}$ , this is the same as

$$\frac{\delta e^{-\frac{x}{mk}} dz}{mk} = (d\omega + d\phi) \cot \omega.$$

That's a bit of a mess, but Euler, ever a genius at substitution, offers us a way to simplify it.

Take  $CZ = z$  and  $CT = t$ . Then  $\sin \omega = \frac{t}{z}$  and  $\cos \omega = \frac{\sqrt{zz - tt}}{z}$ . Taking the differential

gives  $d\omega = \frac{z dt - t dz}{z\sqrt{zz - tt}}$ . Because triangles  $CZT$  and  $Zzu$  are similar and because  $Zu = z d\phi$  and  $uz = dz$ , we

have  $CT : ZT = Zu : zu$ , that is to say  $\frac{t}{\sqrt{zz - tt}} = \frac{z d\phi}{dz}$ , from which we get  $d\phi = \frac{t dz}{z\sqrt{zz - tt}}$ . As a

consequence we have  $d\omega + d\phi = \frac{dt}{\sqrt{zz - tt}}$ . Because  $\cot \omega = \frac{\sqrt{zz - tt}}{t}$ , Euler's messy equation reduces

to

$$\frac{\delta e^{-\frac{x}{mk}} dz}{mk} = \frac{dt}{t},$$

Because  $x = z - a$ , this really involves just two variables,  $z$  and  $t$ , and Euler tells us it is easy to solve.

But Euler doesn't solve it until ten pages later in the paper. Instead, he tells us that there is an easy way to use this form to determine the radius of curvature of at the point  $Z$  and then to use that to answer some of his questions about refraction.

Euler tells us that "as we know", the radius of curvature is  $\frac{z dz}{dt}$ . This seems completely unreasonable to those of us who learned formulas for radius of curvature that involved three-halves powers and second derivatives. However,  $t$  and  $z$  don't form a rectangular or a polar coordinate system. In some of his papers on differential geometry, Euler has shown that for these particular variables, the radius of curvature really is  $\frac{z dz}{dt}$ .

Given this, Euler finds that the radius of curvature at the point  $Z$  is

$$\frac{mkze^{\frac{x}{mk}}}{\delta t} = \frac{mke^{\frac{x}{mk}}}{\delta \sin \omega}$$

Now Euler takes the point  $Z$  to coincide with the point  $A$ , so that  $z = a$ ,  $x = 0$  and  $\omega = \zeta$ . There, the radius of curvature becomes  $\frac{mk}{\delta \sin \zeta}$ . Euler plans to make calculations that do not involve great heights, so he feels safe in assuming that the radius of curvature will not change a great deal between the source of the light at  $O$  and the observer at  $A$ , and he concludes that the curve  $OzZA$  is approximately the arc of a circle of radius  $\frac{mk}{\delta \sin \zeta}$ . He calls this radius  $g$ , then does a bit of calculation, again using *toises* as his unit of length (see last month's column for more about *toises*) and estimating that  $\delta = \frac{3}{10000}$ . He finds that  $g = AG$  is about 13.3 million *toises*, or about 4.08 times the radius of the earth. Fig. 5 is not drawn to scale; the arc  $AD$  is much too large and the radius of curvature  $AG$  is too small.

Euler is ready to use this approximation to do an example. Following Fig. 5, he supposes that the observer at  $A$  sees an object at  $O$  that is apparently exactly on the horizon, so the angle  $\zeta$  is  $90^\circ$ . Let  $OA$  be the path of the ray of light from  $O$  to  $A$  and let  $AG$  be the radius of curvature of that path. Because  $\zeta = 90^\circ$ , we know that  $AG$  passes through the center of the circle.

Further, take angle  $ACO = \varphi$ . Then let  $D$  to be the point on the surface of the earth directly beneath  $O$  and let  $AD = s$ , which makes  $s = a\varphi$ , where  $a = CA$  is the radius of the earth. Now, Euler asks how high  $O$  must be above the point  $D$  for it to be visible to the observer at  $A$ ?

This is a fairly elementary geometry problem that a modern reader might solve using polar coordinates. Euler uses a different strategy, extending the radius  $OC$  to the point  $K$  that makes  $OKG$  a right angle. Then he finds  $OK$  in terms of  $a$ ,  $g$  and  $\varphi$  to be

$$OK = \sqrt{gg \cos^2 \varphi + 2ag \sin \varphi^2 - aa \sin^2 \varphi}.$$

From there, he finds, because  $\varphi$  must be rather small, that  $DO$  is approximately  $\frac{3}{4} \cdot \frac{ss}{2a}$ . All this is fairly routine, so we won't give details.

Euler concludes the main part of this article with a table of vertical adjustments that must be made for observations over various distances from 100 *toises* (about 200 yards) to 40,000 *toises* (almost 50 miles). Then he does an example where the ray of light is elevated from the horizon when it arrives at the observer at the point  $A$ , and finally, almost as an afterthought, he gives an exact solution to the differential equation he had found earlier.

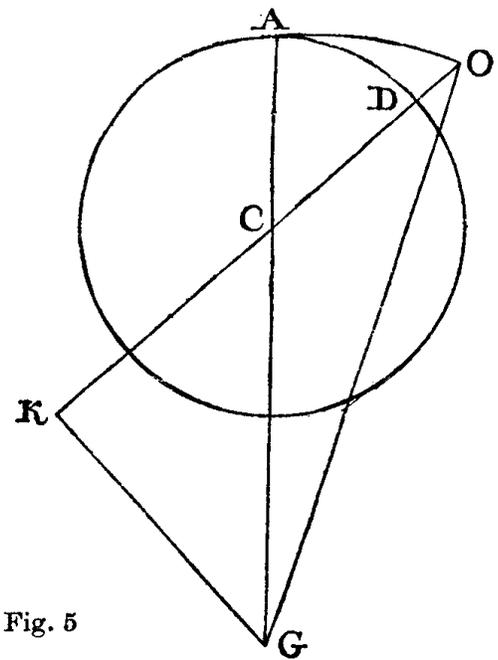


Fig. 5

In this E502, we've seen quite a few aspects of Euler's work in applied mathematics that don't often get a chance to shine in this series of columns. We see how Euler often put a problem down for a while, only to pick it up and extend it more than 20 years later. He had solved a similar problem [E219] in 1754, but he had not studied the optics of the atmosphere in the years in between. We see that even after he became nearly totally blind, Euler still worked on optics. It reminds us of the deaf Beethoven, still composing his Ninth Symphony

On a technical level, we see how Euler makes the transition from the algebraic formulation of physical laws, in this case Boyle's law and Snell's law, to a calculus-based formulation. He was the first to make this a standard technique in science. Euler also shows us how to use good approximations to the trigonometric and exponential functions to get accurate, though not exact, solutions to practical problems.

Euler spent as much time on practical problems that were on the cutting edge of the technological and theoretical frontier of his day as he did on his pure mathematics.

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