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# How Euler Did It

by Ed Sandifer

## Density of air

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Leonhard Euler did an immense amount of work in optics, but that work is not very well known among mathematicians. Seven volumes in Series III of the *Opera omnia* are devoted to Euler's optics, two volumes to his 1769 book the *Dioptricae* and five volumes containing the 56 papers he wrote on the subject. All but six of those papers were published during Euler's lifetime, evidence of how important his work was considered at the time. Several of Euler's *Letters to a German Princess* were devoted to optics as well.

The two volumes of the *Dioptricae* and the 56 papers fill 2522 pages of the *Opera omnia*. By comparison, Euler's work on number theory consists of 96 papers, no books, taking up 1955 pages in four volumes of Series I of the *Opera omnia*.

One of Euler's earliest optics papers is also one of his best known, *Nova theoria lucis et colorum*, "A new theory of light and color" [E88] in which he criticized Isaac Newton's "projectile" theory of light, whereby luminous sources emitted streams of rapidly moving corpuscles and proposed instead that light was rather like sound. Just as sound is a disturbance transmitted by waves in the air, so also, Euler suggested, light is a disturbance in the aether, which Euler and the other scientists of his times believed was a subtle substance filling the universe. [Sandifer Feb 2008] Euler was one of the first to propose a wave theory of light, and his theory was quite influential up until the time of Einstein. [Hakfoort 1995, Home 2007]

The *Dioptrica* is a textbook on optics that leads the reader through the theory of refraction and the function of lenses and how to use the theory to design telescopes and microscopes. Except for being written in Latin and having no problem sets at the end of each chapter, it is much like a modern text on the subject.

In our next two columns, we are going to examine one of Euler's papers on how the Earth's atmosphere refracts light, *Sur l'effet de la réfraction dans les observations terrestres*, "On the effect of refraction on terrestrial observations", [E502] written about 1777 and published in 1780. The paper is quite similar to one of Euler's earlier papers, *De la réfraction de la lumière en passant par l'atmosphere selon les divers degrés tant de la chaleur que de l'élasticité de l'air*, "On the refraction of light passing through the atmosphere due to the different degrees of heat and elasticity of the atmosphere", [E219] written in 1754 and published in 1756. The earlier article applies to celestial observations, for which the light passed through the full depth of the atmosphere. The later article applies to terrestrial observations passing from one altitude to another, say observing a taller mountain from a lower one or observing a ship at sea from a mountain on shore.

Euler begins his article by explaining that,

> "Because of refraction, the stars appear to us higher above the horizon than they really are. We also encounter this same phenomenon in terrestrial observations, where objects always appear higher to us than they would if it were not for refraction. The reason for this is that the rays of light do not always go in straight lines to our eyes, as we ordinarily suppose, but they are found to be a little bit curved, and their concavity is turned downward. … I propose to explain here this effect of refraction and to determine all the phenomena that result from it."

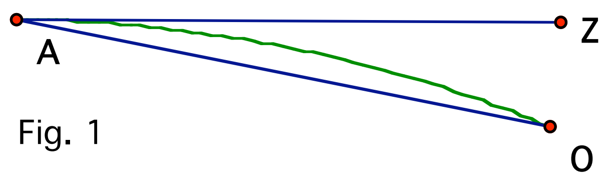


Fig. 1

Euler asks us to look at Fig. 1, showing an object at point *O* from which a ray arrives at point *A* following the curved line *OA*. The direction of this ray at *A* is the straight line *AZ*, which is tangent to the curve at *A*. To the observer's eye, it appears that the object is somewhere along the tangent *AZ* and as a consequence it appears to be higher or taller than it actually is.

From here, Euler's article divides naturally into two parts, the first about how the density of the atmosphere varies with altitude and the second about how that variation on density causes the refraction Euler describes. We divide our two columns along the same lines. In this month's column, we will see how Euler sets up and solves the differential equation that describes the density of the atmosphere as a function of altitude. Next month we will learn about refraction and see how Euler develops tables that correct for that refraction.

Euler tells us that because the curvature of light rays "is caused by the different densities of the atmosphere between the object at *Z* and the eye at *A*, it is necessary to start by determining the law followed by the density of the air as it decreases with altitude." Note that he now locates the object at *Z* instead of at *O*, as he had in Fig. 1. He makes the simplifying assumption that "at equal heights above the surface of the earth, the density of the air is always the same." Because he means to apply this model to line-of-sight observations on the surface of the earth, the errors introduced by this assumption will probably compensate. It will turn out that the *change* in air density will have a bigger impact than the absolute density itself.

In what follows, Euler will use the symbol *q* to denote two different things, the density of air at a point *Q* and also a point infinitely close to that point *Q*. This might be inattentiveness on the part of Euler or his assistants, or he may think that the meaning of the symbol is clear enough from its context and there's no need to introduce a different symbol.

Referring to Fig. 3 (we skip over Fig. 2), Euler takes *c* to be the density of air at *A*, the observer's location. He lets *q* denote the density of air at some point *Q* directly above *A* and he takes *x* to be the height *AQ*. He takes the air pressure at *A*, measured by the height of mercury in a barometer, to be *k* and the pressure at $Q$ to be $p$. Next he cites what we now call Boyle's Law, that the density of the atmosphere is proportional to the air pressure and gets the formula

$$\frac{q}{c} = \frac{p}{k}.$$

Now Euler plans to do calculus. Following his usual 18th century procedures, he takes another point $q$ infinitely close to $Q$ and takes $Qq = dx$. Then the density at $q$ will be $q + dq$ and the height of the barometer, that is to say the air pressure, will be $p + dp$, where, in Euler's words, "it is clear that the differentials $dq$ and $dp$ will have negative values." Now the problem is to find the relations among these three differentials.

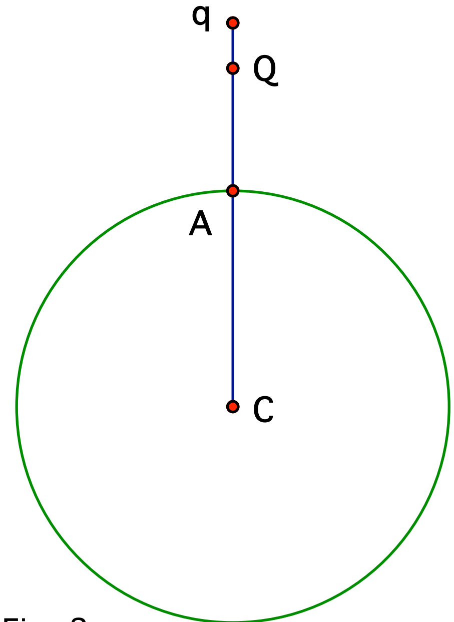

Fig. 3

Because the height of the barometer falls by the quantity $dp$ when the altitude increases by $dx$, it follows that a column of mercury of height $dp$ must weigh the same as a column of air of height $dx$. Euler knew that measurements showed that, near the surface of the earth, mercury is about 10,000 times as dense as air. He denoted this ratio of densities at the altitude of the observer by $m$. At the altitude of the observer, this makes $dx = -m\,dp$.

However, the density of air varies with altitude and the density of air at $Q$ is less than $c$, the density of air at $A$. Thus at $Q$, if the altitude increases by $dx$, the corresponding column of air will be less dense in the ratio of $c$ to $q$. This gives us the differential equation

$$dx = -\frac{mc\,dp}{q},$$

or equivalently,

$$dx = -\frac{mk\,dp}{p}.$$

Euler easily solves the second equation to get

$$x = -mk \ln p + C.$$

When $x = 0$ we have $p = k$ so $0 = -mk \ln k + C$. This tells us that $C = mk \ln k$, thus $x = -mk \ln p + mk \ln k$. We can rewrite this last formula either as

$$x = mk \ln \frac{k}{p} \text{ or as } x = mk \ln \frac{c}{q}.$$

Euler was writing for an 18th century European audience, so he gives the values of these constants in units that would have been familiar to his readers. The value of $k$, he says, is about 28 *pouces* or $2\frac{1}{3}$ *pieds de Paris*. Today, these are about 30 inches, 750 mm or 1000 millibars. He adds that the product $mk$ expresses a height of about 23,333 *pieds* or 4000 *toises*. A *toise* was a French unit of measure, usually equal to six feet, but sometimes six and a half, and the length of a foot varied across different parts of France. The unit became obsolete with the establishment of the metric system after the French revolution of 1789, but people continued to use it for many years in Louisiana and Quebec. Euler's *toises* seem to be about 6.25 of our modern feet, so 4000 *toises* are about 25,000 in today's feet.

Euler notes that his equation can be rewritten in a number of forms, starting with $\frac{x}{mk} = \ln\frac{k}{p} = \ln\frac{c}{q}$. He can exponentiate to get either $\frac{c}{q} = e^{\frac{x}{mk}}$ or $\frac{q}{c} = e^{-\frac{x}{mk}}$. This last version expands into a Taylor series approximation to give

$$\frac{c}{q} = 1 - \frac{x}{mk} + \frac{xx}{mmkk} - \frac{x^3}{m^3k^3}.$$

He uses the second version because it leads to an alternating series and he knows that it will converge more quickly.

Euler tells us that for all but the highest mountains, $x$ is considerably less than $mk$, so even this approximation converges very quickly, and that for most observations only the first two terms are necessary, so it is usually accurate enough to take

$$\frac{q}{c} = 1 - \frac{x}{mk}.$$

With this, Euler has a good formula giving $q$, the density of air in terms of $c$, the density at the altitude of the observer and $x$, the altitude above the observer, as well as in terms of known or observable constants.

Euler opens the next section of his paper with the words, "Having determined the density of air for all heights above the level of the level of the observer at the point $A$, it remains to determine the refraction suffered by a ray of light as it passes from air of one density into air of another density." This will require another set of differential equations that incorporate Snell's law, and will be the subject of next month's column.

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*How Euler Did It* is updated each month.