

	<h1>How Euler Did It</h1> <p>by Ed Sandifer</p>	
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Nearly a cosine series

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To look at familiar things in new ways is one of the most fruitful techniques in the creative process. When we look at the Pythagorean theorem, for example, and we ask, “what if it’s *not* a right triangle?” it leads us to the Law of Cosines. When we ask, “what if -1 *did* have a square root?” we discover complex variables. To find variations on a familiar theme isn’t the only tool in our creative repertoire, but it is one of the most reliable.

But it doesn’t always work. This month we’ll look at a variation on the Taylor series for $\cos x$. It seems like a pretty good idea, and Euler does his best to make something of it, but after a few promising results, it just doesn’t go anywhere.

The Taylor series for $\cos x$ is very familiar:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdots 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdots 8} - \text{etc.}$$

Indeed, combining it with the corresponding Taylor series for $\sin x$ and for e^x , then taking x to be an imaginary number, $i\theta$, leads to one of the most popular ways to prove the Euler identity, $e^{i\theta} = \cos\theta + \sqrt{-1} \sin\theta$, and its famous special case, $e^{\pi i} = -1$. [Sandifer August 2007]

Sometime late in 1776 or early in 1777, as he neared the age of 70, Euler looked at this series and asked what he would get if he changed the denominators. Euler had asked such questions before. Note that the denominator on the term of degree m is what we now write $m!$ and call “ m -factorial.” Early in his career, when he was only 22 years old, he had discovered what we now call the Gamma function by asking what would happen if m were a fraction instead of a whole number, and this had been a very productive line of inquiry, as we have seen in previous columns. [Sandifer September 2007, Sandifer October 2007, Sandifer November 2007]

Instead of changing the number of factors in the products, Euler wondered what would happen if he changed where the products begin, say starting at a number n instead of starting with 1. That is to say, he asked about the series

$$1 - \frac{x^2}{n(n+1)} + \frac{x^4}{n(n+1)(n+2)(n+3)} - \frac{x^6}{n \cdots (n+5)} + \text{etc.}$$

Euler doesn't give this series a name. Usually, he just specifies a series by stating a value of n . We will usually do that as well, but when we want to compare the series for two different values of n , we'll distinguish them by using the notation $\cos_n x$.

Obviously, if $n = 1$, this is the same as the Taylor series for $\cos x$, that is $\cos_1 x = \cos x$. After a moment's reflection we notice that the series is undefined if $x = 0$ or if x is any negative integer, but what about other values of n , larger integers or fractions?

Euler knows that there are no bounds and few rules about how to *be* creative, but that once you have an idea, especially in mathematics, you should pursue it methodically. Euler begins by looking at the cases $n = 1$, $n = 2$ and $n = 3$.

As we saw, $\cos_1 x = \cos x$, but Euler takes note of its important properties in case they become part of a pattern that persists for larger values of n . For him, the most important property is that the roots form an arithmetic sequence, extending in the positive and the negative direction. The smallest positive root is $\frac{\pi}{2}$ and the difference between consecutive roots is π .

In the case $n = 2$, the series is

$$1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^6}{2 \cdots 7} + \frac{x^8}{2 \cdots 9} - \text{etc.}$$

This is the Taylor series for $\frac{\sin x}{x}$, one of Euler's favorites because of the crucial role it played in his solution to the Basel problem some four decades earlier. Except for a gap at $x = 0$, the roots of this formula are all multiples of π , positive and negative. They too form an arithmetic sequence.

Moving on to $\cos_3 x$, where $n = 3$, we get the series

$$1 - \frac{x^2}{3 \cdot 4} + \frac{x^4}{3 \cdot 4 \cdot 5 \cdot 6} - \frac{x^6}{3 \cdots 8} + \text{etc.}$$

Euler tells us that *evidens est seriei propositae summam esse*, "it is obvious that the given series sums to be"

$$\frac{2(1 - \cos x)}{x^2}.$$

It wasn't that obvious to me, but it's true. It follows from multiplying the given series by $\frac{x^4}{1 \cdot 2}$. I'll leave out the details and hope that the reader is as amused by filling them in as I was.

Again with the exception of $x = 0$, this has roots every time $\cos x = 1$, that is whenever x is a non-zero multiple of 2π . And again, the roots form an arithmetic sequence. There seems to be a pattern!

So we move with confidence to the case $n = 4$. The series is

$$1 - \frac{x^2}{4 \cdot 5} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} - \frac{x^6}{4 \cdots 9} + \text{etc.}$$

Readers who enjoyed checking Euler's claim for the case $n = 3$ might also enjoy discovering that this series sums to

$$\frac{6(x - \sin x)}{x^3}.$$

We see that when $x = 0$, this equals 1, either the hard way (a triple application of l'Hôpital's rule) or the easy way (substitute $x = 0$ into the series itself). Because $x = 0$ is the only root of the numerator, we see that $\cos_4 x$ has no roots at all. The pattern is broken!

What happened? Euler tries to explain it by noting that in the case $n = 3$, all the roots were double roots, and they were twice as far apart as they had been in the case $n = 2$. He says that, as in the case of polynomials, a pair of real roots will "coalesce" into a double root as they are about to become imaginary roots. He also claims a small consolation because the function goes to zero as x goes to infinity.

Disappointed, but undeterred, Euler prepares to study the cases where n is a fraction. For any value of n , he defines s by the equation

$$\frac{s}{x^{n+1}} = \cos_n x.$$

This makes

$$s = x^{n-1} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+3}}{n(n+1)(n+2)(n+3)} - \text{etc.}$$

Now Euler is going to use his calculus tricks on the series. Introduce a new function z by taking

$$s = x^{n-1} - z$$

so that

$$z = \frac{x^{n+1}}{n(n+1)} - \frac{x^{n+3}}{n \cdots (n+3)} + \frac{x^{n+5}}{n \cdots (n+5)} - \text{etc.}$$

Now take the second derivative of this with respect to x and get

$$\frac{\partial \partial z}{\partial x^2} = x^{n-1} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+3}}{n \cdots (n+3)} - \text{etc.} = s,$$

from which we have

$$\frac{\partial \partial z}{\partial x^2} + z = x^{n-1}.$$

Now, Euler tells us, “the whole business (*totum negotium*) reduces to the solution of this second degree equation.”

Alas, Euler can’t solve this equation either, so this approach turns out to be a dead end. According to Maple™, this differential equation has a solution in terms of Lommel functions, which are supposedly related to Bessel functions. But Euler couldn’t know that because Lommel wrote in the 1880s, a hundred years after Euler died.

Euler tries something else. In the case $n = \frac{1}{2}$, we get

$$\cos_{1/2} x = 1 - \frac{4x^2}{1 \cdot 3} + \frac{16x^4}{1 \cdot 3 \cdot 5 \cdot 7} - \frac{64x^6}{1 \cdot 3 \cdots 11} + \text{etc.}$$

To find the roots, Euler sets this equal to zero and makes the substitution $z = 4xx$, where we should note that this z is a new variable not the same as the function z that appeared in the differential equations above. So, Euler wants to solve

$$0 = 1 - \frac{z}{1 \cdot 3} + \frac{zz}{1 \cdots 7} - \frac{z^3}{1 \cdots 11} + \frac{z^4}{1 \cdots 15} - \text{etc.}$$

Thwarted in his analytical efforts to solve it, he resorts to numerical methods. He observes that this is an alternating series with decreasing terms. A number of years earlier, [E212, E247, Sandifer June 2006] Euler had developed a technique for accelerating the convergence of such a series. In general, if we write a general alternating series with decreasing terms as

$$1 - a + b - c + d - e + \text{etc.}$$

and if we take

$$\begin{aligned} \alpha &= 1 - a \\ \beta &= 1 - 2a + b \\ \gamma &= 1 - 3a + 3b - c \\ &\text{etc.} \end{aligned}$$

then the sum is given by

$$\frac{1}{2} + \frac{\alpha}{4} + \frac{\beta}{8} + \frac{\gamma}{16} + \text{etc.}$$

Euler uses this to estimate that $z = 4.20$, hence $x = 1.025$.

Euler checks his work with a similar but more accurate method he says is due to Daniel Bernoulli and estimates that $z = 3.31$ so $x = .909$, significantly smaller than $\frac{\pi}{2} \approx 1.571$, which is the smallest positive root of $\cos_{\frac{1}{2}} x$. Euler studies at the ratio between .909 and 1.571. He does a continued fraction expansion and decides that the ratio is likely to be $1 : \sqrt{3}$, and suspects that the root of $\cos_{\frac{1}{2}} x$ that he has been pursuing is probably $x = \frac{\pi}{2\sqrt{3}}$, or 0.90695. This is an astonishing bit of guesswork on Euler's part.

Euler can carry this no farther and he does not try to find the second or third roots of $\cos_{\frac{1}{2}} x$. Instead, he moves on to $n = \frac{1}{4}$. This time he goes straight to Bernoulli's method and estimates that the smallest root of $\cos_{\frac{1}{4}} x$ is $x = 0.5717$, which he guesses is $\frac{\pi}{\sqrt{30}}$, or about 0.57356.

The case $n = \frac{1}{3}$ leads to an estimate that $x = .6875$, a value that Euler says is not easily related to π , as the other two values had been.

At this point, Euler must be a bit discouraged. He gives up on chasing roots for fractional powers of n without even noting that as n gets smaller, the value of the smallest root seems to get smaller as well. He spends a few more pages trying to identify some of the complex roots that arise when $n > 3$, and is unable to find relations that describe them. Despite the paucity of his results, Euler ends his article with the remark, "Still, I believe that by extending this argument, in which several excellent ingenuities occur, Geometry would not be displeased."

Euler was satisfied with this paper, and hoped that someone could add to it. That took a hundred years, after the development of the idea of special functions, an idea for which Euler planted the seeds, but did not live to seem them sprout and flourish.

References:

- [E212] Euler, Leonhard, *Institutiones calculi differentialis*, St. Petersburg, 1755, reprinted as *Opera omnia* Series I vol. 10. English translation of Part I by John Blanton, Springer, NY, 2000.
- [E247] Euler, Leonhard, De seriebus divergentibus, *Novi commentarii academiae scientiarum Petropolitanae* 5 (1754/55) 1760, pp. 205-237. Reprinted in *Opera omnia* Series I vol. 12, pp. 585-617. Also available online at EulerArchive.org.
- [E684] Euler, Leonhard, De radicibus aequationis infinitae $0 = 1 - \frac{x^2}{n(n+1)} + \frac{x^4}{n(n+1)(n+2)(n+3)} - \frac{x^6}{n \cdots (n+5)} + \text{etc.}$, *Nova acta academiae scientiarum Petropolitanae* 9, (1791), 1795, pp. 19-40, 166-169. Reprinted in *Opera omnia* I.16a, pp. 241-265. Available online at EulerArchive.org.
- [Sandifer June 2006] Sandifer, Ed, Divergent Series, *How Euler Did It*, MAA Online, June 2006. Reprinted in *How Euler did it*, pp. 177-184, MAA, Washington, DC, 2007.
- [Sandifer August 2007] Sandifer, Ed, e , π and i , *How Euler Did It*, MAA Online, August 2007.
- [Sandifer September 2007] Sandifer, Ed, Gamma the function, *How Euler Did It*, MAA Online, September 2007.
- [Sandifer October 2007] Sandifer, Ed, Gamma the constant, *How Euler Did It*, MAA Online, October 2007.
- [Sandifer November 2007] Sandifer, Ed, Inexplicable functions, *How Euler Did It*, MAA Online, November 2007.

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