



How Euler Did It



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Saws and modeling

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Euler seemed to be interested in everything, and when he was interested in something, he sought to understand it with mathematics. Somehow, he got interested in saws, and in 1756, while working at the Berlin Academy, he wrote a 25-page paper, *Sur l'action des scies*, "On the action of saws" [E235].

I can only speculate on why Euler decided to write this paper. In their Editors' Introduction to the volume of the *Opera omnia* where this paper is reprinted, Charles Blanc and Pierre de Haller suggest that "its main purpose was to show one more time the possibility of putting into play the laws of mechanics and the techniques of mathematical analysis for measuring out the best advantage for concrete situations. One does not find in fact any truly new idea about the use of mathematics in practice."

I would suggest a more practical and timely motivation. When Euler first arrived in St. Petersburg in 1728, his actual position was as Physician to the Russian Navy. He mostly worked at the St. Petersburg Academy, but the Navy had some claim on his time and efforts. Reportedly, as part of a tour of military facilities, Euler visited some of the sawmills that provided lumber to the Navy. There he learned both of the critical importance of lumber to military operations and of the actual operation of sawmills.

Almost 30 years later, Euler was working in the academy of the Prussian King Frederick II, who was about to embark on what is often called the Seven Years War, except in America where we learn of it as the French and Indian War. It would also make sense if Euler, remembering the strategic importance of lumber supplies, were trying to help his King in the forthcoming war effort.

Regardless of his motivations, Euler begins his analysis of saws by describing the scope of his problem. First, he is interested in vertical saws being moved by a machine in a steady, repeatable way, not in horizontal saws or saws that are being moved by the hands of real people. He has two reasons for this restriction. First, the actual motion of people is hard to describe, and second, this reflects the design of actual sawmills.

The machines he studies will lift the saw blade and advance the timber, and then the saw blade will fall under its own weight, making the cut as it descends. Such saws were used for centuries to make the long cuts that slice trees into boards. Euler wanted to calculate the best values for the various

measurements in the design of the saw and blade, the blade's length, its number of teeth, the width and depth of those teeth, and the number of men necessary to operate the saw most efficiently.

Euler describes a saw ABCD, as shown in Fig. 1, attached to move constantly along a vertical line EF, alternately ascending, when it does no work, and descending under its own weight. There are a lot of parts in this system, and he starts with the smallest one he can think of, the action of a single tooth as it cuts into the wood to a certain depth. He tells us that the resistance to this action depends on:

- 1. the hardness of the wood,
- 2. the size of the tooth, and
- 3. the depth to which it penetrates the wood.

For this third item, the depth ought be neither too large nor too small, for if it is too small, the tooth won't cut any wood, and if it is too large the tooth won't go through the wood. Euler uses ρ to denote the resistance to the tooth, but he doesn't give us any particular units for that resistance. Further, he lets α be the depth of the cut, item 3 on his list, and he assumes, perhaps from experience or from experiment, but more likely from some thought experiment, that ρ is proportional to the square of α .

It is clear, at least to Euler, that each tooth of the saw should act equally on the wood. Thus, each tooth should penetrate to the same depth and the teeth should not be arranged parallel to AC, for then the first tooth would do all the cutting and the rest of them would not touch the wood. Hence the line of teeth should be slanted, being farther from AC at C than at A, and the teeth should be arranged in arithmetic sequence, first tooth at k, next at $k+\alpha$ (where α is the depth of each cut), etc.

Taking the length of the saw AC = f, the number of teeth to be *n*, and the depth of each penetration to be α , he calculates the angle for the saw to be ζ , where

$$\tan\zeta = \frac{\left(n-1\right)\alpha}{f} \, .$$

Solving this for α gives

$$\alpha = \frac{f \tan \zeta}{n - 1}$$

so the difference between CD and AB will be

$$n\alpha = \frac{nf\tan\zeta}{n-1}$$

Now let the total number of teeth be v and let the interval PQ be the part of the saw actually touching the wood. Call the length of that interval z. The resistance the wood makes on the saw, taking into account z, the thickness of the wood and f, the total number of teeth in the saw, works out to be



 $\frac{nz\rho}{f}$. Euler introduces the parameter $R = n\alpha$, which represents what the resistance would be if all the

f teeth were in contact with the wood at once, and rewrites the actual resistance as $\frac{z}{f}R$. Because

 $R = n\rho = \frac{f\rho \tan \zeta}{\alpha}$, Euler rewrites it again as

$$\frac{z\rho \tan \zeta}{\alpha}$$

To end his description of the saw blade, Euler takes c to be the depth to which one complete stroke of the saw cuts. Then $c = n\alpha \approx f \tan \zeta$. This last formula is only an approximation because of the nf tan ζ way Euler measured ζ above when he wrote $n\alpha$

$$e \ n\alpha = \frac{ny \ cany}{n-1}$$

So far, Euler has used ten different, though related measurements to describe his saw and its action: α , ρ , k, n, ζ , f, z, R and c. He is now ready to consider the motion of the saw as it is descending. The saw is supposed to be attached to a weight heavy enough so that the saw will fall under its own weight without any other force being applied. He plans to calculate the total time of the descent of the saw.

Euler decides that the saw will fall from a height *a* at the beginning to build up speed before the teeth start to cut the wood. In the 18th century, it was difficult to measure small intervals of time accurately, and different places used different units of length. To account for this, Euler took as his standard length a unit g, the distance through which a falling body will fall in one second. Then he tells us that the free fall will last $\sqrt{\frac{a}{g}}$ seconds. We would do that calculation a bit differently today, and we would use the symbol g to be the acceleration due to gravity, not the (closely related) distance it represents here.

Speeds were also difficult to measure accurately. Euler and his contemporaries used a similar idea to describe speed. They measured it as the height from which a freely falling body would have to fall to achieve the same speed. We will see this principle in action a little later.

Euler divides the rest the motion of the saw as it descends into three phases. The first phase lasts from when the first saw tooth at B reaches the wood until it exits the other side. During this interval, the resistance to the saw steadily increases as more and more teeth cut into the wood. The second phase lasts from the end of the first phase until the last saw tooth at D reaches the wood. During this second phase, the number of teeth touching the wood is constant, so the resistance to the saw is constant as well. Finally, the third phase lasts from the end of the second phase until the last saw tooth at D exits the other side of the wood, at which time the saw should stop falling, lest its energy be wasted. Euler calculates the times for each of these three phases.

For the first phase, Euler refers to Fig. 2, where GH = MN = b is the thickness of the wood and GB = z is the length of the part of the saw in contact with the wood instead of the thickness of the wood as it was before. During the first phase, z increases from 0 to b. Let us consider the action at the moment that z has some particular value. Take v to be the speed of the saw at that instant, measured, as we mentioned above, as the distance from which a freely falling body must drop to achieve that speed.

Let P be the total weight of the saw. Following the analysis of the saw that Euler did above, the resistance to the saw at that point will be

$$\frac{z\rho\tan\zeta}{\alpha} = \frac{c\rho}{\alpha}\cdot\frac{z}{f}.$$

Take $N = \frac{\rho \tan \zeta}{c}$, so that the resistance can be simplified to Nz. Euler tells us that now "the principles of Mechanics furnish us with the following equation:"

$$Pdv = (P - Nz)dz$$
 or $dv = dz\left(1 - \frac{Nz}{P}\right).$

This is F = ma in disguise. Integrating this and using the initial condition v = a when z = 0, we get

$$v = a + z + \frac{Nzz}{2P}$$

Thus, at the end of the first time interval, when the tooth B arrives at the point H, we have z = b, and the speed of the saw corresponds to a freefall from the height $a + b - \frac{Nbb}{2P}$.

Knowing the speed at every point z, Euler does a difficult calculation to find the time that elapes in the first phase. He introduces a new constant, $e = \frac{P}{N}$, not to be confused with that other constant *e* that he sometimes used. This makes

$$v = a + z - \frac{zz}{2e} = \frac{2ae + 2ez - zz}{2e}.$$

Now, because of the way Euler and his friends measured time and speed, the quantity Euler denoted by v, is not the same as velocity, the quantity we denote by v today. For our v, we have $\frac{dz}{dt} = v$, but, confusingly, for Euler's v, we have instead $\frac{dz}{dt} = \sqrt{v}$. Given this, the previous expression becomes

$$dt = \frac{dz\sqrt{2e}}{\sqrt{2ae + 2ez - zz}}.$$

When we integrate this from z = 0 to z = b, we find that the duration of the first phase is

$$\frac{\sqrt{e}}{\sqrt{2g}} \left(\sin^{-1} \frac{e}{\sqrt{2ae + ee}} - \sin^{-1} \frac{e - b}{\sqrt{2ae + ee}} \right) \text{ seconds.}$$

Similar analyses with correspondingly similar figures lead us to find that the duration of the second phase will be

 ${\mathcal B}$

 \mathcal{H}

Fig. 2

A

м

$$\frac{\sqrt{2e}}{2(e-b)\sqrt{g}}\left(\sqrt{2ae+bb+2(e-b)f}-\sqrt{2ae+2be-bb}\right) \quad \text{seconds},$$

and that of the third will be

$$\frac{\sqrt{e}}{\sqrt{2g}} \ln \left(\frac{e + \sqrt{2ae + 2be + 2f(e - b)}}{e - b + \sqrt{2ae + bb + 2f(e - b)}} \right) \text{ seconds.}$$

Thus, the total time it takes the saw to descend is the sum of these three durations plus the interval of free fall at the beginning. Euler writes it out as a formidable sum. We omit it here.

By now, the King, for whom I propose Euler was writing, has probably given up and stopped reading this paper. That's too bad, because Euler is finally going to draw some conclusions. In particular, all of the quantities involved in this sum have to be real, so that none of the quantities under the radical signs can be negative. Euler does not mention that the argument of the logarithm function cannot be negative or zero, and the quantities of which we take the arcsine must be between -1 and +1, so he only gets two conditions:

I.
$$a+b > \frac{bb}{2e}$$
 and
II. $a+b > \frac{1}{2}e + f\left(\frac{b}{e}-1\right)$.

Unless these two conditions are satisfied, the saw will stop part way through its descent. This might happen because the tree is too thick (b is too large) or if the saw is too long (f is too large). The saw could also stop if the weight P is too small or the parameter N, which measures the resistance of the wood and the size of the teeth, is too large. Cryptic as these formulas may be, they furnish actual constraints on the design of the saw itself. Euler provides several pages of discussion on what these formulas mean and how to use them.

Finally, Euler turns to personnel issues. Apparently, for the saws that he has seen, the saw blade is lifted by one or several men working, rather than by the action of a water wheel or by teams of animals, as sometimes shown in old pictures. Euler supposes that there are *m* men working to lift the saw, and that each of them lifts the saw through a distance *s* each second, and that each of them can lift a weight of *S* pounds. He calculates that the *m* men working together can lift the saw at a rate of $\frac{mSs}{P}$ feet per second. The total distance the saw must be lifted is a+b+f feet, where still *a* is the distance of free fall at the beginning of the saw's descent, *b* is the thickness of the wood and *f* is the length of the saw. Given this, it will take the *m* men $\frac{P(a+b+f)}{mSs}$ seconds to lift the saw the required a+b+f feet.

Euler pauses to confess some of the errors of approximation that he knows he is making. He argues that he can neglect what he hopes is a small distance that the saw goes past the bottom of the piece of wood. He knows that he is also ignoring friction and the effort necessary to advance the tree the distance c in each cycle of the saw.

To begin an example, take *T* to be the total time of one cycle of the saw, adding the time it takes to lift the saw to the time he found earlier for the saw to descend. If a = 0, b = 1, e = 1 and f = 3, all values that describe the saw itself, then g = 15.625, and he finds that

$$T = 0.79643 + \frac{4P}{mSs}$$
 seconds.

Let's think about how big this saw is. It has a three-foot blade, and it is cutting a tree that is a foot thick, so the action of the saw takes the top end of the saw to a height of 7 feet above the ground. Euler is about to assume that the saw itself weighs P = 189 pounds, so it appears that Euler's saw, though fairly large and heavy, is still small enough that it could be moved around on a cart or a wagon.

He makes some further guesses about the men operating the saw and the weight of the saw itself. He supposes that s = 2 feet per second and that S = 30 pounds. Given this, he estimates the amount that various numbers of men can saw in one hour:

1 man	3.82 feet per hour
2	7.23
3	10.30
4	13.06
5	15.57

Euler notes that there are diminishing returns here, and that infinitely many men could saw 67.5 feet of timber an hour. He verifies this phenomenon with a few more examples and concludes that "it is advantageous to employ as small a number of men as possible."

He further notes that a three-foot saw is far more efficient than a two-foot saw, and gives the advice that the saw ought to be as long as possible as well. He suggests four feet.

With this, Euler ends the paper rather abrubtly.

Euler's many books and papers range from the entirely abstract to the eminently practical. It seems to me that his most abstract work, especially that in number theory, he did for the sheer joy and beauty of the mathematics, but that almost all the rest of it was rooted in the hope that it would have practical applications in the real world. He only wrote a few papers that were as explicitly concerned with the mechanics of machinery as this one, but some of those were quite important and involved difficult mathematics.

The best known of these is his work on the strength of columns, where he combines the calculus of variations and the theory of elastic curves to calculate the buckling strength of columns and to design columns with the greatest possible strength. He also wrote on the shape of gear teeth, energy from windmills, on earthen dams and on caring for winter wheat crops. There seems to be no end to his interests or to his confidence in the applications of mathematics.

References:

 [[]E235] Euler, Leonhard, Sur l'action des scies, Mémoires de l'académie des sciences de Berlin 12 (1756) 1758, pp. 267-291. Reprinted in Opera omnia Series II vol. 17, pp. 66-88. Also available online at EulerArchive.org.

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