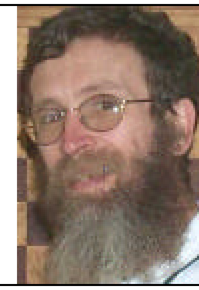
How Euler Did It

by Ed Sandifer



Curves and paradox

October 2008

In the two centuries between Descartes (1596-1650) and Dirichlet (1805-1859), the mathematics of curves gradually shifted from the study of the means by which the curves were constructed to a study of the functions that define those curves. Indeed, Descartes' great insight, achieved around 1637, was that curves, at least the curves he knew about, had associated equations, and some properties of the curves could be revealed by studying those equations. Almost exactly 200 years later, in 1837, Dirichlet gave his famous example of a function defined on the closed interval [0, 1] that is discontinuous at every point, namely

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

The roles of the two objects had been reversed and Mathematics had become far more interested in the study of functions than in the study of curves.

Euler came roughly half way between Descartes, both in years and in the evolution of mathematical ideas. He was instrumental in the early development of the modern idea of a *function* used the concept to lead mathematics away from its geometric foundations and replace them with analytic, i.e. symbolic manipulations, but he could not foresee how general and abstract the idea of a function could eventually become.

In 1756, Euler was devoting much of his intellectual powers to using differential equations to study the world. As we saw last month, he used them to model fluid flow. A future column will be devoted to how he used differential equations to design more efficient saws. It should be no surprise that he also used differential equations get new results about curves.

When Euler wrote *Exposition de quelques paradoxes dans le calcul integral*, (Explanation of certain paradoxes of integral calculus), [E236], he posed four problems with a distinctly 17th century flavor, as we shall see. Then he used his new tools of differential equations to solve the problems. He dwelt as much on his clever technique for solving the problems as he did on the geometry itself. Indeed, he was so intrigued by his technique that he dubbed it a "paradox" and used that word in the title of his article. In this column, we will look at the first of his four problems, see why I've described it as having a 17th century flavor, and then look at Euler's clever solution.

Euler describes the problem as follows:[1]

Problem 1: *Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere.* (*Fig. 1*)

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

This is a little confusing, and the problem is neither the clarity of Euler's French nor the quality of Andrew Fabian's translation. Both are excellent. Rather, it is that some of the vocabulary of geometry has changed in the last 250 years. Let us untangle it as we go along.

First, let A be the given point, and take the line AP to be an axis. The curve we want to find is EM, and a little later on, Euler will assume that m is another point on the same curve infinitely close to M. The special property that Euler wants his curve to have involves the line MV, which is tangent to the curve EM at the point M. He wants the distance between this line MV and the given point A to be the same, for every line tangent to the curve EM.

Before proceeding to Euler's answer to his own question, let's see why we described this problem as having the flavor of the century before Euler.

In ancient times, Euclid studied some of the properties of the lines tangent to a circle. His main result was that the tangent lines are perpendicular to their corresponding radii. The next century, Apollonius determined the lines tangent to a parabola by determining the point at which the tangent to a particular point intersects the axis of the parabola.

There was little new in the world of tangents until the early 1600s, when Descartes and Fermat (1601-1665) each found algebraic ways to find things we now recognize as being equivalent to tangents. In each case, the object they used was either a line segment or the length of a line segment. In Fig. 2, let TN be an axis and let EMm be a curve. If the line TM is tangent to the curve EMm at the point M and if T is the point where that tangent intersects the axis, then the segment MT, or its length, is what Descartes

[1] Here and elsewhere, when we quote Euler's article, we use the fine translation by Andrew Fabian.

or Fermat would have called the *tangent*. The projection of the tangent onto the axis, that is the segment *PT*, was called the *subtangent*. The other two objects involved the line perpendicular to the curve *EMm* at the point *M*. Taking *N* to be the point where that perpendicular intersects the axis, the segment *MN* was the *normal* and its projection onto the axis, that is the segment *PN*, was the *subnormal*.

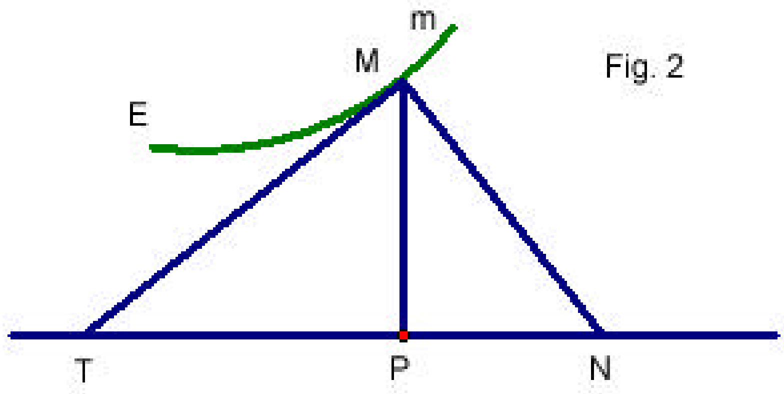


Fig. 2

Note that as modern readers, we easily recognize these line segments that are closely related to the derivative of a function at a point, but in the 1600s, hardly anyone had taken even Calculus I, so recognizing this relationship would have been much more difficult for them.

One of the founders of the Paris Academy, Claude Perrault, (1613-1688) recognized a physical problem related to the length of the tangent segment. Let us imagine a boy walking along the axis *AP* and pulling a wagon, assumed to be not on the axis, with a rope of fixed length *a*. People knew enough about resolution of forces to know that the direction the wagon moved would always be the same as the direction of the rope, and because the length of the rope was constant and the boy stays on the axis, the rope itself forms the segment we called the Tangent. Hence, the curve traced by the wagon is the curve for which the length of the tangent segment is always equal to the fixed length *a*. Perrault named this curve the *tractrix*, and to find an equation for the tractrix became one of the important unsolved problems of the late 1600s.

Another important problem was posed by Florimond de Beaune, (1601-1652) one of the people who helped Frans van Schooten (1615-1660) translate Descartes' *Geometrie* into Latin and then to write commentary that helped other scientists understand it better. De Beaune asked to find a curve for which the subtangent had a fixed length *a*. De Beaune did not give this curve a name, but we now know it to be an exponential curve.

Christian Huygens (1629-1695) solved the problem of the tractrix in 1693, and G. W. Leibniz (1646-1716) solved de Beaune's problem in 1684 his first paper on calculus, "Nova Methodus pro Maximis et Minimis."

There are two other obvious problems in this same vein, to find curves for which the normal segments have a fixed length, and for which the subnormal segments have a fixed length. Though these are both easily found using calculus, it turns out that their solutions do not *require* calculus, so whoever solved them first didn't become famous for their solutions.

Now we can re-phrase Euler's Problem 1 in the language of the 1600s: to find a curve *EM* for which every tangent segment *MV* passes a fixed distance *a* from the origin *A*.

As usual, Euler begins his solution of the problem by assigning variables. Again, see Fig. 1. He takes the line *AP* as his axis, and takes the length $AP = x$. The corresponding ordinate is $PM = y$. From the description of the problem, we know that $AV = a$, where AV is perpendicular to the tangent line *MV*. Euler adds calculus by introducing an infinitesimal arc element $Mm = ds$. Then the corresponding changes in x and y are $dx = Pp = Mp$ and $pm = dy$, where the segment *Mp* is taken to be parallel to the axis *AP*. Moreover, all this makes $ds = \sqrt{dx^2 + dy^2}$.

Euler plans to use similar triangles, so he needs to build some more triangles. He introduces a new segment, *SP*, perpendicular at *S* to the tangent line *VM*, and a second segment *AR*, perpendicular to *SP* at *R*. Note that this makes $a = AV = RS = PS - PR$.

We have three similar triangles, *DPMS*, *DAPR* and the differential triangle *DMmp*, and ratios of corresponding parts give us

$$PS = \frac{Mp \cdot PM}{Mm} = \frac{y\,dx}{ds} \quad \text{and} \quad PR = \frac{mp \cdot AP}{Mm} = \frac{x\,dy}{ds}.$$

Substituting this into our observation that $a = PS - PR$, we get

$$y\,dx - x\,dy = a\,ds = a\sqrt{dx^2 + dy^2}.$$

Euler tells us that this equation *exprimera la nature de la courbe cherchée*, "will express the nature of the curve being sought."

He sets out to solve this problem, squaring both sides and performing a well-choreographed sequence of substitutions. It is messy. Three pages and seventeen display equations later, Euler leads to see that the differential equation has an infinite family of solutions

$$y = \frac{n}{2}(a + x) + \frac{1}{2n}(a - x),$$

where n is an arbitrary constant, and the equation also has a singular solution, a family of one, namely

$$xx + yy = aa.$$

Readers familiar with differential equations will recognize that it is not a coincidence that the lines in the infinite family are the tangent lines to the circle $xx + yy = aa$. Note that Euler has overlooked the vertical line that corresponds to the case $n = 0$.

So, Euler has solved the problem, and it was a little bit interesting, but there seems to be nothing to suggest the "paradox" mentioned in the title of the article. We must read on to learn what Euler found paradoxical in this problem.

He continues by offering us an easier way to solve this same problem. He begins his second solution by introducing a new function, p, defined by the equation

$$dy = p\,dx.$$

A modern reader immediately recognizes p as the first derivative, but in Euler's day, the fundamental tools of differential calculus were differentials, and when they needed what we now call a derivative function, they had to define it as a quotient of differentials. The modern concept and notation was introduced by Lagrange in 1797. This definition of p makes

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+p^2}$$

in general, and it lets us rewrite the equation that "expresses the nature of the curve" as

(1) $$y - px = a\sqrt{1+pp} \quad \text{or} \quad y = px + a\sqrt{1+pp}.$$

Note that they differentials seem to have disappeared here, though they are actually hiding in the definition of the variable p.

Here comes the "paradox." "[I]nstead of integrating this differential equation, I differentiate it" and get

$$dy = p\,dx + x\,dp + \frac{ap\,dp}{\sqrt{1+pp}}.$$

The "paradox" in this was more dramatic to Euler's readers because in his time, chapters on differential equations often bore titles like "On the *integration* of equations" rather than "On solving differential equations." Differential equations were ones that one resolved by integration, and the word "solving" was reserved for easier problems.

So, now we know what the paradox is. How does it work? We have $dy = p\,dx$, and subtracting this from the previous equation gives

(2) $$0 = x\,dp + \frac{ap\,dp}{\sqrt{1+pp}}.$$

First, divide this by dp (later, we'll need to remember that we divided by dp) and solve for x to get

(3) $$x = -\frac{ap}{\sqrt{1+pp}}.$$

Then substitute this into equation (1) to get

(4) $$y = -\frac{app}{\sqrt{1+pp}} + a\sqrt{1+pp} \quad \text{or} \quad y = \frac{a}{\sqrt{1+pp}}.$$

Viewed as a differential equation, equation (4) would take a good deal of work to solve, but viewed in conjunction with equation (3), we see a parameterization of the curve being sought in terms of the parameter p. We have

$$x = -\frac{ap}{\sqrt{1+pp}} \quad \text{and} \quad y = \frac{a}{\sqrt{1+pp}}.$$

Squaring these and adding them together gives

$$xx + yy = \frac{aapp + aa}{1+pp} = aa,$$

the equation of a circle, which is one of the solutions to the given problem.

But what about all those other solutions, the infinite family of straight lines? This method does not seem to provide us with them. But let's take a closer look at equation (2), and recall that we divided it by dp. In doing so, we may overlook solutions corresponding to the cases when $dp = 0$. In Euler's words, equation (2) also "contains" the solutions $dp = 0$. This would mean that p is a constant, and Euler chooses to call that constant n. Substituting $p = n$ into equation (1) gives us the infinite family of lines lines,

$$y = nx + a\sqrt{1+nn}.$$

With this, Euler has solved his differential equation by differentiating, rather than by integrating. Euler has a good deal more to say about other geometrically inspired problems that also lead to "paradoxical" differential equations, but for this column, this one will have to suffice. I hope that this account has whetted the readers' appetites for such problems, because Hieu Nguyen, professor and sometime chair of Mathematics at Rowan University in Glassboro, New Jersey and his student, Andrew Fabian have translated all of E236 from French into English and made it available through The Euler Archive. They are preparing an article for publication that describes E236 in considerably more depth than we have here, and have found a plethora of fascinating insights and new problems. Hieu Nguyen spoke on their results at the 2008 annual meeting of The Euler Society when they met in New York in July.

It is part of the magic of mathematics in general and of the works of great mathematicians like Euler in particular, that so often there are new things to be found in old mathematics. Watch for their article in the not-too-distant future.

Special thanks to Andrew Fabian for his English translation of E236 and to Hieu Nguyen for drawing this paper and Andrew Fabian's translation to my attention.

References:

[E236] Euler, Leonhard, Exposition de quelques paradoxes dans le calcul integral, *Mémoires de l'académie des sciences de Berlin*, **12**, (1756), 1758, pp. 300-321. Reprinted in *Opera omnia*, Series I vol 22, pp. 214-236. Available online, both in its original and in English translation by Andrew Fabian, at EulerArchive.org.

[Lagrange 1797] Lagrange, Joseph-Louis, *Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toute considération d'infiniment petits et d'évanouissans de limites ou de fluxions, et réduits l'analyse algébrique des quantités finies*, Imprimérie de la République, Paris, An V (1797). Available online at Gallica.bnf.fr.

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How Euler Did It is updated each month.