

	<h1>How Euler Did It</h1> <p>by Ed Sandifer</p>	
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## Life and Death – Part 2

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Last month we began this two part series on Euler's work in actuarial science with an account of his study of mortality, the "death" part of "Life and Death." This month we turn to the other half of the equation and ask the mathematical question, "Where do those babies come from?"

To seek his answers, Euler begins with a number of assumptions. Some of them are just to simplify the beginnings of his analysis and will be replaced with more sophisticated assumptions later. Others are simple because he sees no way to gather the data to support more complex ones. Still others are just naïve.

For notation, Euler takes  $M$  to be the current population and, taking both births and deaths into account, he takes  $mM$  to be the population one year later. He patiently explains that if births and deaths are equal in number, then  $m = 1$  and the population will remain the same, that if births exceed deaths then  $m > 1$  and the population will increase, etc. This level of detail is unusual, even for Euler. Clearly he expects that some of the people who read this paper do not know higher mathematics.

Euler presents his first assumption on the birthrate, what he call he "multiplication," as follows:

"Now, having fixed the principle of propagation, which depends on marriages and fertility, it is evident that the number of infants which are born in the course of a year ought to have a certain ratio to the number of living men."

Though Euler himself was the father of 13 children, he was also a man of the 18th century, and like his contemporaries, would have thought it unseemly to mention any role *women* might play in the propagation of the species, other than the oblique reference contained in the word "fertility" (*fécondité*). He also primly and properly assumes that all children are born inside of wedlock, an assumption as untrue then as it is now, at the same time being irrelevant to the mathematics of his model. Perhaps he was just being hopeful, as he had two teenage daughters when he wrote this paper.

Having made this assumption, Euler sets out to avoid using it. He says that he could just take the number of births,  $N$ , to be some constant multiple of  $M$ , say  $\alpha M$ , where  $\alpha$  is the measure of fertility, "[b]ut it is difficult to draw from this the consequences about birth rates and other phenomena that depend on it." He doesn't give details.

Instead, paralleling his notation on the population itself, he takes  $N$  to be the number of births at present and  $nN$  to be number of births in one year. If we take  $n = \alpha m$ , then this would be consistent with Euler's assumption on birth rates. Instead of calculating  $n$ , though, Euler hopes to observe it. He notes that the number of births each year form a geometric progression, increasing or decreasing depending on whether  $n > 1$  or  $n < 1$ .

Now, Euler combines this result with his results from the first half of E334, those that we presented in last month's column. We remind the reader that  $(m)$  represents the proportion of a population of infants still alive after  $m$  years. This said, he gives us a table:

	Number of births	After 100 years there are still living
at present	$N$	$(100)N$
after 1 year	$nN$	$(99)nN$
after 2 years	$n^2N$	$(98)n^2N$
after 3 years	$n^3N$	$(97)n^3N$
$\vdots$	$\vdots$	$\vdots$
after 98 years	$n^{98}N$	$(2)n^{98}N$
after 99 years	$n^{99}N$	$(1)n^{99}N$
after 100 years	$n^{100}N$	$n^{100}N$

The numbers in this last column form what we would now call the "age distribution" of the population, a concept that plays a key role in modern demographics and population dynamics. The sum of the numbers in the last column gives the population after 100 years, namely

$$(1) \quad n^{100}N \left( 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.} \right).$$

Because people are mortal, this is a finite series.

For this and the rest of his analysis to be accurate, it is important that these mortality figures, (1), (2), (3), etc., as well as the value of  $n$  be stable, and that they have been stable long enough that the age distribution becomes stable as well. Euler will make this disclaimer at the end of the article.

Now Euler takes the current population to be  $M$ , the births per year to be  $N$  and claims that

$$(2) \quad \frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}$$

He gives only sketchy reasons, but we can fill in some of his steps. If we take  $M_{100}$  to be the population in 100 years, then  $M_{100}$  is given by formula (1). From the previous table, he number of births in 100 years will be  $n^{100}N$ . Since the ratio between population and the number of births is taken to be constant, we get that

$$\frac{M}{N} = \frac{M_{100}}{N_{100}} = 1 + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.},$$

as claimed.

For Euler, this is an important result because it allows him to calculate  $n$  in terms of  $M$  and  $N$ . The value of  $n$  was difficult to observe directly and was very sensitive to small counting errors, so it was more accurate to calculate it indirectly.

With these theoretical tools in hand, Euler raises, then answers some questions, much like he did in the first half of the paper.

**Question 1.** – *Given the hypotheses of mortality and fertility, and if we know the population, to find how many people there are of each age.*

In modern terms, Euler seeks the age distribution we mentioned above. If we multiply both sides of formula (2) by  $N$ , the resulting formula gives the total population  $M$  as a sum of the population at age, namely  $N$  infants,  $\frac{(1)}{n}N$  people of age 1,  $\frac{(2)}{n^2}N$  people of age 2, ..., and in general  $\frac{(a)}{n^a}N$  people of age  $a$ .

**Question 2.** – *Given the same things, to find the number of men who die in a year.*

Take  $M$ ,  $N$  and  $n$  as before, and note that  $\frac{M}{N} = \frac{1}{n}$ . Then in a year the population will become  $nM$ , so the change in population will be  $nM - M$ . The number of births will be  $nN$ , and the rest of the population change will be accounted for by deaths, so the number of deaths must be  $(1 - n)M + nN$ .

**Question 3.** – *Knowing how many births and funerals happen during the course of a year, to find the total population and its annual growth, under a given hypothesis of mortality.*

This was a particularly interesting question in Euler's time, to estimate the total population from the numbers of births and deaths, both of which were thought to be easily and accurately available. Note that the "given hypothesis of mortality" of which Euler speaks means the values of those mortality fractions, (1), (2), (3), etc.

Let  $N$  be the number of births, as always, and  $O$  be the number of funerals. Question 3 asks us to find  $M$  given  $O$ ,  $N$  and the "given hypothesis of mortality." From Question 2 we have

$$(3) \quad O = (1 - n)M + nN.$$

so that

$$M = \frac{O - nN}{1 - n}.$$

From this, it is clear that we will have to use the "given hypothesis of mortality" either to find  $n$  or to eliminate it from the equation. Towards this end, a bit of algebra gives

$$\frac{M}{N} - 1 = \frac{O - N}{N(1 - n)}.$$

Recall also formula (2):

$$(2) \quad \frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}$$

Substituting this into the preceding formula gives

$$(4) \quad \frac{O - N}{N(1 - n)} = \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \text{etc.}$$

Now Euler unnecessarily divides the problem into three cases, a stable population, an increasing one, and a decreasing one.

In the first case, the number of births equals the number of deaths, so formula (3) implies that  $n = 1$ . (Mathematically, we have to admit that perhaps  $M = N$ , but that cannot happen in reality.) In the case  $n = 1$ , formula (2) gives

$$M = N(1 + (1) + (2) + (3) + (4) + \text{etc.}).$$

In the second case, if  $N$ , the number of births is greater than  $O$ , the number of deaths, then  $N - O$  is positive and the population is increasing and that  $n > 1$ . Likewise, if the number of deaths exceeds the number of births, then the population decreases and  $n < 1$ .

It seems that Euler has done an incomplete, or at best an evasive job of answering Question 3, for after reading his answer, we still don't know how to find the population  $M$  given the numbers of births and deaths,  $N$  and  $O$  and a "given hypothesis of mortality." In order to provide a number for an answer, we must find a value for  $n$ . We're not allowed to use the fact that  $n = \frac{N}{M}$  because we don't know  $M$ ; that's what the question asks us to find. Instead, we have to solve formula (4) for  $n$ , but that is a polynomial of degree 100, and it is likely to be difficult to solve. Euler only tells us whether  $n$  is greater than or less than one, and not how to find an actual value. We will come back to this in Question 5.

As if he has answered Question 3, Euler asks:

**Question 4.** – *Given the numbers of births and deaths in a year, to find how many of each age there will be among the dead.*

Euler takes  $M$ ,  $N$ ,  $O$  and  $n$  as before, and assumes that we are given  $N$  and  $n$ . Then he solves his problem with a series of tables. His first table uses the birth rate,  $n$ , this year's number of births,  $N$ , and the "given hypothesis of mortality" to calculate the age distributions for this year and next year.

Number	this year	next year
of newborns	$N$	$nN$
of age one year	$\frac{(1)}{n}N$	$(1)N$
of age two years	$\frac{(2)}{n^2}N$	$\frac{(2)}{n}N$
of age three years	$\frac{(3)}{n^3}N$	$\frac{(3)}{n^2}N$
etc.		etc.

From this we can calculate the number of each age who die each year:

	number of deaths
less than one year	$(1 - (1))N$
1 to 2 years	$((1) - (2))\frac{N}{n}$
2 to 3 years	$((2) - (3))\frac{N}{n^2}$
3 to 4 years	$((3) - (4))\frac{N}{n^3}$
4 to 5 years	$((4) - (5))\frac{N}{n^4}$
etc.	etc.

This table answers Question 4, but Euler wants to take it just a little farther. In this table, the sum of the entries in the second column must be the total number of deaths,  $O$ . Making that sum and dividing by the common factor  $N$  gives

$$\frac{O}{N} = 1 - (1)\left(1 - \frac{1}{n}\right) - \frac{(2)}{n}\left(1 - \frac{1}{n}\right) - \frac{3}{n^2}\left(1 - \frac{1}{n}\right) - \text{etc.}$$

This, he notes, agrees with formulas (2) and (3) above.

**Question 5.** – *Knowing the number of living people as well as the number of births and the number of deaths of each age over the course of a year, to find the law of mortality.*

Solving formula (3) for  $n$  gives

$$n = \frac{M - O}{M - N}.$$

This gives a much easier way to find  $n$  than the method suggested in Euler's Question 3, though not necessarily cheaper. This way we have to find  $M$ . This means spending the time and effort to take a

census of all the people living in the city or state. Note though that if  $M$  is enough larger than  $O$  and  $N$  or if  $O$  and  $N$  are relatively close to each other, small errors in their measurement don't have much effect on  $n$ , so perhaps the census doesn't have to be extremely accurate.

Continuing with the problem at hand, let  $\alpha, \beta, \gamma, \delta$ , etc. be the number of deaths at ages less than one year, one to two years, two to three years, etc. This gives values to the entries in the second table from Question 4. Then, from the first line of that table, we get

$$\alpha = (1 - (1))N,$$

so that

$$(1) = 1 - \frac{\alpha}{N}.$$

Likewise

$$\beta = ((1) - (2))\frac{N}{n},$$

so

$$\begin{aligned} (2) &= (1) - \frac{n\beta}{N} \\ &= 1 - \frac{\alpha + n\beta}{N}. \end{aligned}$$

Continuing in this way,

$$\begin{aligned} (3) &= 1 - \frac{\alpha + n\beta + n^2\gamma}{n}, \\ (4) &= 1 - \frac{\alpha + n\beta + n^2\gamma + n^3\delta}{n}, \end{aligned}$$

and the pattern is evident.

This is a remarkable result, in my mind the best in the whole article. Euler recognizes clearly, and says as much in his closing paragraphs, that mortality and fertility vary a great deal from one area to another, and that it would be impractical to gather the information that went into Kersseboom's tables for very many locations. However, they routinely kept track of births and deaths, or at least of baptisms and funerals. Euler's Question 5 shows that the information in Kersseboom's tables, though difficult to gather directly, can be recovered from other information that is much easier to collect.

We should note that some of these results can be found using modern tools. We can combine the "law of mortality" with age-specific fertility rates and building a transition matrix to describe how many babies are born to people of various ages and what proportion of people of each age survive to be the next age. Then quantities like Euler's survival rates, (1), (2), etc., the values of  $n$  and  $\alpha$  and the age distribution can be expressed in terms of eigenvalues, eigenvectors and such. The subject provides an early example in many mathematical modeling courses and a late one for linear algebra courses.

A wise man once said, "You can't always get what you want." You may want mortality tables like Kersseboom's, they are difficult and expensive to prepare. Euler shows us that, with careful analysis and good mathematical modeling, that the wise man was correct when he added, "but if you try sometime, you just might find you can get what you need."

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References:

- [E334] Euler, Leonhard, Recherches générales sur la mortalité et la multiplication du genre humain, *Mémoires de l'académie des sciences de Berlin* **16** (1760) 1767, pp. 144-164. Reprinted in *Opera omnia* I.7, pp. 79-100. Available online, along with a link to an English translation by Richard Pulskamp, at EulerArchive.org.
- [E335] Euler, Leonhard, Sur les rentes viagères, *Mémoires de l'académie des sciences de Berlin* **16** (1760) 1767, pp. 165-175. Reprinted in *Opera omnia* I.7, pp. 101-112. Available online, along with a link to an English translation by Richard Pulskamp, at EulerArchive.org.
- [Kersseboom 1748] Kersseboom, William, *Essays on Political Arithmetic contained in Three Treatises*, The Hague, 1748. This includes an English translation of Kersseboom's 1742 mortality tables.
- [Lewin 2003] Lewin, C. G., *Pensions and Insurance Before 1800: A Social History*, Tuckwell Press, East Lothian, Scotland, 2003.

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