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How Euler Did It

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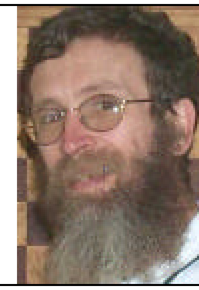

Life and Death – Part 1

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The history of mortality tables and life insurance is sprinkled with the names of people more famous for other things. American composer Charles Ives, who, like your columnist, worked in Danbury, Connecticut, also invented the insurance agency, so that insurance customers themselves no longer had to negotiate directly with the insurance companies. Edmund Halley, of comet fame, devoted a good deal of energy to calculating one of the earlier mortality tables. [Halley 1693] Henry Briggs, better known for his pioneering work with logarithms, calculated interest tables. Swiss religious reformer John Calvin preached that life insurance was not necessarily immoral usury, as some maintained at the time. Daniel Defoe, author of *Gulliver's Travels*, proposed a national insurance scheme for England in 1697. We find other familiar names, DeMoivre, Fermat, Harriot, Hudde, Huygens, de Witt, van Shooten, Maclaurin, Maseres, Pepys and, of course, Euler.

Euler wrote half a dozen articles that related to mortality and life insurance, and several other articles about lotteries and card games that used many of the same principles of probability. His first excursion into this particular subject was in 1760, when he had been working in Berlin for almost 20 years. Euler was probably inspired to write *Recherches générales sur la mortalité et la multiplication du genre humain*, (General research on mortality and population growth of human kind) [E334] by his colleague at the Berlin Academy, Johann Peter Süssmilch, who published a book on population and mortality tables in 1761.

Süsssmilch's interest was in the spirit of the times. Many European cities, including Berlin, had recently taken censuses. In 1747, Berlin took two censuses, the first counting 107,224 residents, and the second differing by less than 200. [Lewin 2003] In 1755, they counted 126,661. In 1748, Euler put some examples in his *Introductio in analysin infinitorum* [E101] to show that observed populations in the hundred thousands and millions were not inconsistent with a world-wide human population of just two at the time of Biblical Creation, assuming only modest growth rates. Many of the names mentioned above as contributing to the history of such tables were contemporaries of Euler, and it is not surprising that he joined in.

Euler was clearly also aware of a mortality table published by Willem Kersseboom (1691-1771) in 1742. Kersseboom's table, reproduced along with the *Opera omnia* edition of E334, gives the number of survivors from among an initial population of 1400 newborns after x years, for x going up to 95 years old. The first few lines of Kersseboom's table are reproduced below.

Age x	Number of survivors l_x	Age x	Number of survivors l_x	Age x	Number of survivors l_x	Age x	Number of survivors l_x
0	1400						
1	1125	11	886	21	808	31	699
2	1075	12	878	22	800	32	687
3	1030	13	870	23	792	33	675
4	993	14	863	24	783	34	665
5	964	15	856	25	772	35	655

Euler uses this same idea, measuring the proportion of survivors among a hypothetical population of N infants born at the same time. He denotes by (1) the proportion of survivors after one year, (2) the proportion after two years, (3) after three years, etc., and he estimates that (125) must be less than one in 100 million. Initially, for the sake of generality Euler assumes only that (1), (2), (3), … is a decreasing sequence of fractions between zero and one. Later in the paper Euler assigns particular observed values to these proportions, a few of which are:

(1) = 0.804	(11) = 0.633	(21) = 0.577	(31) = 0.499
(2) = 0.768	(12) = 0.627	(22) = 0.571	(32) = 0.490
(3) = 0.736	(13) = 0.621	(23) = 0.565	(33) = 0.482
(4) = 0.709	(14) = 0.616	(24) = 0.559	(34) = 0.475
(5) = 0.688	(15) = 0.611	(25) = 0.552	(35) = 0.468

Except for some round-off errors, Euler's proportions correspond exactly to Kersseboom's data.

Having established his notation, Euler explains how to use it. He starts with a simple example, to calculate how many of these people will die each year. The gives us a small table:

From	0	years to	1	years there die	$N-(1)N$.
"	1	"	2	"	$(1)N-(2)N$.
"	2	"	3	"	$(2)N-(3)N$.
"	3	"	4	"	$(4)N-(3)N$.
"	4	"	4	"	$(5)N-(4)N$.

Euler probably gave us this example just to make sure we understand his notation, for he does not count it among the six questions he now sets out to answer:

Question 1. – *A certain number of men being given, all of whom are the same age, to find how many will probably still be alive a certain number of years later*.

Euler's analysis is clear and direct. He takes the number of men to be M and their age to be m. Then the initial number of men is determined by

$$M=(m)N,$$

so

$$N=\frac{M}{(m)}.$$

Thus, n years later, these men will be aged $m + n$, so the number of them still alive will be $(m+n)N$, or, in terms of M,

$$\frac{(m+n)}{(m)}M.$$

Perhaps to prepare us for questions about mortality and about probability that he will be asking later, Euler adds as a remark that the number of these men that we expect will die over the next n years will be

$$\left(1-\frac{(m+n)}{(m)}\right)M.$$

Question 2. – *To find the probability that a man of a certain age will still be alive a certain number of years later*.

Let m be the age of the man. From what we learned in answering Question 1, we know that the proportion of men of that age who will be alive n years later is

$$\frac{(m+n)}{(m)},$$

so that is the probability that the man will still be alive in n years.

Continuing his practice of giving us more of an answer than the question asked for, and because Euler grew up among Bernoullis and he knows the basics of probability very well, Euler adds that the probability that the man will die during the next n years is

$$1-\frac{(m+n)}{(m)}.$$

He also points out that these probabilities become equal when $(m + n) = ½\,(m)$. He describes the value of n that makes this true as he number of years for which the man's "hope of survival" equals his "dread of death."

Question 3. – *We ask the probability that a man of a certain age will die in the course of a given year*.

Using the techniques Euler has just taught us, it is easy to calculate that the probability that a man of age m will die in n years, that is to day that he will die at the age of $m + n$, is just

$$\frac{(n)-(n+1)}{(m)}.$$

Again, Euler answers more than we asked, and tells us that the probability that the man of age m will die between n years and $n + v$ years from now is

$$\frac{(n)-(n+v)}{(m)}$$

and the probability that he will die on a particular date n years from now is

$$\frac{(n)-(n+1)}{365(m)}.$$

Question 4. – *To find the term in which a man of a given age can hope to survive, so that it is equally probable that he will die before this term as after*.

This question is curious, for Euler has already answered it in his remarks on Question 2. Here, though, he lets z be the age to which the man can hope to survive, that is the quantity he denoted by $m + n$ in Question 2. The solution is still the value of z that makes $(z)=\frac{1}{2}(m)$. Euler says that we call the interval $z - m$ the *power of life* (*la force de la vie*) of a man aged m years.

Up to this point, all of Euler's questions have been more curiosity than practicality. Now he turns to questions of money and the practical problem of assigning a value to what we call *life annuities*, or in French, *rentes viageres*. A life annuity is a contract to pay a fixed amount of money to a person every year until that person dies. Life annuities tend to be worth more for a younger person (though not always) because the younger person is likely to live longer. In Euler's terms, the younger person usually has a higher power of life. Also, the time value of money (usually) makes money worth more today than the same amount of money will be worth at some future date. Euler has both of these complications in mind when he asks

Question 5. – *To determine the life annuity that it is fair to pay each year until his death to a man of any age, in exchange for an amount that he pays in advance*.

Euler tells us the age of the man, m, and the amount he pays in advance, a, and asks us to suppose that there are M such men and that x is the amount that the life annuity will pay each year. He also makes the unstated assumption that there will be no payments during the first year, so after one year the number of men surviving will be $\frac{(m+1)}{(m)}M$, and the total amount of all the life annuities that must be paid in one year will be

$$\frac{(m+1)}{(m)}Mx.$$

Similarly, the total amount to be paid after two years will be

$$\frac{(m+2)}{(m)}Mx,$$

and after three years it will be

$$\frac{(m+1)}{(m)}Mx, \quad \text{etc.}$$

Next, Euler introduces the time value of money. He explains that a sum money S payable in n years at 5 percent interest has a present value of only $\left(\frac{20}{21}\right)^n S$. In the interests of generality, he takes λ

to be the annual growth rate, 1.05 in the case of 5% interest, and he notes that the value $\frac{20}{21}$ corresponds to $\frac{1}{l}$.

It is time for another table.

	they ought to pay	which is presently worth
after 1 year	$\frac{(m+1)}{(m)}Mx,$	$\frac{(m+1)}{(m)}\cdot\frac{Mx}{l},$
after 2 years	$\frac{(m+2)}{(m)}Mx,$	$\frac{(m+2)}{(m)}\cdot\frac{Mx}{l^2},$
after 3 years	$\frac{(m+3)}{(m)}Mx,$	$\frac{(m+3)}{(m)}\cdot\frac{Mx}{l^3}$
	etc.	etc.

Fairness requires that the value of what is paid out, that is the sum of the entries in the last column of Euler's table, be equal to the amount paid in. This, and a tiny bit of algebra, makes

$$a = \frac{x}{(m)}\left(\frac{(m+1)}{l} + \frac{(m+2)}{l^2} + \frac{(m+3)}{l^3} + \frac{(m+4)}{l^4} + \text{etc.}\right),$$

so that

$$x = \frac{(m)a}{\frac{(m+1)}{l} + \frac{(m+2)}{l^2} + \frac{(m+3)}{l^3} + \frac{(m+4)}{l^4} + \text{etc.}}.$$

This answers Question 5, though the calculations involved in actually evaluating the formula for given values of m and l are quite tedious. In his next paper on the subject [E335] Euler gives some techniques to shorten the calculations, and based on the Kersseboom tables and a 5% interest rate, gives tables of the fair prices for life annuities of 100 *écus* for all ages.

Next, Euler turns to his last and most practical question, one that applies to annuities as they were actually sold in Euler's time:

Question 6. – *When the annuities are for newborn babies and the payments do not begin until they have attained a certain age, to determine the amount of these payments.*

This financial instrument roughly corresponds to a modern trust fund, established at the birth of a child but not making payments until that child reaches some age or status, say age 21 or enrolling in college.

The same techniques that answered Question 5 serve us well here, and Euler omits most of the work, jumping straight to the conclusion that

$$a = x\left(\frac{(n)}{l^n} + \frac{(n+1)}{l^{n+1}} + \frac{(n+2)}{l^{n+2}} + \frac{(n+3)}{l^{n+3}} + \text{etc.}\right),$$

where n denotes the year when the first payment is to be made. Solving for x we get

$$x = \frac{a}{\frac{(n)}{l^n} + \frac{(n+1)}{l^{n+1}} + \frac{(n+2)}{l^{n+2}} + \frac{(n+3)}{l^{n+3}} + \text{etc.}}$$

Some readers will have noticed that Euler refers to the lives of *men* throughout. This is not entirely 18th century sexism. Euler knew from reading the works of Kersseboom and Struyck [Kersseboom 1748, Struyck 1740] that men and women had different patterns of morality, so Euler's results really did apply to *men* and not necessarily to women. They also were aware that mortality was different in the cities than in the countryside and that it varied among countries and climates. Euler doesn't mention it, but other authors also knew that their data was distorted to some degree by migration from the country to the city in a way that made city-dwellers seem longer-lived than their country cousins.

This brings us to the end of the first half of E334, the part of the paper about mortality. It provides a natural stopping point. We plan to return to the second half of this paper next month, when we will see what Euler has to say about "multiplication of human kind," that is to say population growth.

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