How Euler Did It

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A Series of Trigonometric Powers

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The story of Euler and complex numbers is a complicated one. Earlier in his career, Euler was a champion of equal rights for complex numbers, treating them just like real numbers whenever he could. For example, he showed how to integrate $\int \frac{1}{x^2+1}dx$ without using inverse trigonometric functions. He factored $x^2+1=\left(x+\sqrt{-1}\right)\left(x-\sqrt{1}\right)$, then used partial fractions to rewrite

$$\frac{1}{x^2+1}=\frac{\frac{1}{2}\sqrt{-1}}{x+\sqrt{-1}}-\frac{\frac{1}{2}\sqrt{-1}}{x-\sqrt{-1}},$$

then integrated this difference to get

$$\int \frac{1}{x^2+1}dx=\tfrac{1}{2}\sqrt{-1}\ln\frac{x+\sqrt{-1}}{x-\sqrt{-1}}.$$

Euler typically omitted constants of integration until he needed them, and also seldom used *i* in place of $\sqrt{-1}$. His role in that particular notational innovation is exaggerated.

Euler struck a second, and better-known blow for justice for complex numbers when he took the variable in the exponential function e^x to be an imaginary number, say $x=q\sqrt{-1}$, and showed that

$$e^{q\sqrt{-1}}=\cos q+\sqrt{-1}\sin q.$$

Euler continued to use complex numbers late in his life, but his applications seem to me to be less sweeping and more technical, showing how they solved a variety of specific problems. This month we look at one such problem from 1773.

The title of E447 is "Summatio progressionum $\sin j^{l}+\sin 2j^{l}+\sin 3j^{l}+K+\sin nj^{l}$, $\cos j^{l}+\cos 2j^{l}+\cos 3j^{l}+K+\cos nj^{l}$." Right off, this is confusing to the modern reader, because

Euler writes $\sin j^{\,l}$ where we would write $\sin^l j$ and mean $(\sin j)^l$. For this, we will use the modern notation.

Euler begins by asking us to let

$$\cos j + \sqrt{-1}\sin j = p \quad \text{and}$$
$$\cos j - \sqrt{-1}\sin j = q.$$

Then, from de Moivre's formula, we have

$$\cos nj = \frac{p^n + q^n}{2} \quad \text{and}$$
$$\sin nj = \frac{p^n - q^n}{2\sqrt{-1}},$$

and because $\sin^2 j + \cos^2 j = 1$, we have

$$pq = 1.$$

Properties of geometric series tell us

$$p^a + p^{2a} + p^{3a} + \text{L} + p^{na} = \frac{p^a\left(1 - p^{na}\right)}{1 - p^a}$$

and

$$q^a + q^{2a} + q^{3a} + \text{L} \;\; q^{na} = \frac{q^a\left(1 - q^{na}\right)}{1 - q^a}.$$

If we add these together and repeatedly apply the identities $pq = 1$ and $p^{ka} + q^{ka} = 2\cos ka j$ (a consequence of de Moivre's formula), we get

$$-1 + \frac{\cos na j - \cos(n+1)a j}{1 - \cos a j}.$$

Likewise, if we subtract the q-series from the p-series we get

$$\frac{\sin a j - \sin(n+1)a j + \sin na j}{1 - \cos a j}\sqrt{-1}.$$

Euler uses an integral sign, ?, where we would use a summation sign, ?, so he writes these results as

(1)
$$\int \left(p^{na} + q^{na}\right) = -1 + \frac{\cos na j - \cos(n+1)a j}{1 - \cos a j} \quad \text{and}$$
$$\int \left(p^{na} - q^{na}\right) = \frac{\sin a j + \sin na j - \sin(n+1)a j}{1 - \cos a j}\sqrt{-1}.$$

Now Euler is ready to work on the sums in the title of the article. He takes $l = 1$, and his two series become

$$s = \sin j + \sin 2j + \sin 3j + \mathrm{L} + \sin nj$$
$$= \int \sin nj \quad \text{and}$$
$$t = \cos j + \cos 2j + \cos 3j + \mathrm{L} + \cos nj$$
$$= \int \cos nj \,.$$

Because of de Moivre's identities,

$$\sin nj = \frac{p^n - q^n}{2\sqrt{-1}} \quad \text{and}$$
$$\cos nj = \frac{p^n + q^n}{2},$$

these two series can be rewritten as

$$2s\sqrt{-1} = \int \left(p^n - q^n\right) \quad \text{and}$$
$$2t = \int \left(p^n + q^n\right)$$

But from formula (1) above, and taking $a = 1$, this gives

$$s = \frac{\sin j + \sin nj - \sin(n+1)j}{2(1 - \cos j)} \quad \text{and}$$
$$t = -\frac{1}{2} + \frac{\cos nj - \cos(n+1)j}{2(1 - \cos j)}.$$

Note how unexpectedly simple these formulas are. They each the sum of n terms using only the terms at the beginning and the terms at the end, without using any of the terms in between.

Now take $l = 2$ so that

$$s = \sin^2 j + \sin^2 2j + \mathrm{L} + \sin^2 nj$$
$$= \int \sin^2 nj \quad \text{and}$$
$$t = \cos^2 j + \cos^2 2j + \mathrm{L} + \cos^2 nj$$
$$= \int \cos^2 nj \,.$$

Recalling that $pq = 1$ we get

$$\begin{aligned}\sin^2 nj &= (\sin nj)^2 \\ &= \left(\frac{p^n - q^n}{2\sqrt{-1}}\right)^2 \\ &= \frac{p^{2n} - 2p^n q^n + q^{2n}}{-4} \\ &= \frac{1}{2} - \frac{p^{2n} + q^n}{4}.\end{aligned}$$

Similarly,

$$\cos^2 nj = \frac{1}{2} + \frac{p^{2n} + q^{2n}}{4}.$$

Summing these, we get

$$4s = 2\int 1 - \int \left(p^{2n} + q^{2n}\right) \quad \text{and}$$

$$4t = 2\int 1 + \int \left(p^{2n} + q^{2n}\right)$$

Obviously, $\int 1 = n$, so, using formula (1) we get

$$s = \frac{n}{2} + \frac{1}{4} - \frac{\cos nj - \cos 2(n+1)j}{4(1 - \cos 2j)} \quad \text{and}$$

$$t = \frac{n}{2} - \frac{1}{4} + \frac{\cos nj - \cos 2(n+1)j}{4(1 - \cos 2j)}.$$

It is reassuring to note that $s + t = n$, as it should be, because s is a sum of n squared sines and t is a sum of the corresponding squared cosines.

Euler does $l = 3$ and $l = 4$, and his expressions for s and t grow first to three, then to four terms, though the terms grow no more complicated, except for involving higher powers of 2. Moreover, his expressions have the same general form.

Let's look a bit more closely at this expression for s in the case $l = 2$, the sum of the squares of a sequence of sines. Note how the last term does not increase as the number n increases. Also, if $\cos 2j$ is not very close to 1, then the denominator in the last term is not very small. Moreover, the two cosines in the numerator are always between -1 and 1, so their difference is between -2 and 2. Consequently, the last term is bounded between two values, $+M$ and $-M$, that do not depend on n. Hence, s is always between $\frac{n}{2} + \frac{1}{4} + M$ and $\frac{n}{2} + \frac{1}{4} - M$. Thus, as n goes to infinity, so also does s. The same reasoning applies to t.

Note that this was not the case for the series corresponding to $l = 1$. The last terms in the expressions for s and t are both bounded, by the same argument we gave above, but neither expression contains a term that goes to infinity as n increases.

Indeed, these remarks about $l = 1$ are true for all odd exponents. That is to say, if l is odd, then neither *s* nor *t* increase without bound as *n* increases, but for l even, the series behaves like $l = 2$, and both *s* and *t* grow without bounds.

Euler notices this, too, and wants to examine it a bit. In Euler's time, there were several notions of the *value* of a series. One of them, proposed by Jakob Bernoulli, was that the value was the limit of the average value of the partial sums. Using this notion, the series

$$1-1+1-1+1-1+1-\text{etc.}$$

would have value equal to ½, because half the time the partial sums are 1 and half the time they are zero. Hence, the weighted average value of the partial sums is ½.

This is apparently the notion that justifies Euler's next steps. Taking $l = 1$, he has shown that

$$\begin{aligned} s &= \sin j + \sin 2j + \sin 3j + \text{L} + \sin nj \\ &= \frac{\sin j + \sin nj - \sin(n+1)j}{2(1-\cos j)}. \end{aligned}$$

Euler argues that the average value of $\sin nj - \sin(n+1)j$ is zero, so, if we let *n* go to infinity the value of the now-infinite series *s* can be considered to be

$$s = \frac{\sin j}{2(1-\cos j)}.$$

The same analysis makes the infinite series

$$t = -\frac{1}{2}.$$

Modern analysts throughout the world cringe at this, because Euler has given an exact, finite sum to two series for which the terms do not converge to zero. The analysts don't let us do that anymore.

Perhaps Euler realizes we may have doubts about this particular result, for he reassures us that it is easy to show that this makes sense. He rewrites *t* as

$$t = \frac{\cos j - 1}{2(1-\cos j)}.$$

Multiplying both sides by $2 - 2\cos j$ and writing *t* as the series it represents, Euler gets

$$\begin{aligned} \cos j - 1 &= (2-2\cos j)t \\ &= (2-2\cos j)(\cos j + \cos 2j + \cos 3j + \text{L}) \\ &= 2\cos j + 2\cos 2j + 2\cos 3j + 2\cos 4j + \text{etc.} \\ &\quad - 2\cos^2 j - 2\cos j \cos 2j - 2\cos j \cos 3j - \text{etc.} \end{aligned}$$

Now, from the angle addition formula for cosines we know that in teneral

$$2\cos a \cos b = \cos(a-b)\cos(a+b).$$

Applied to the negative terms of the preceding series, this makes

$$\begin{aligned}
2\cos^2 j &= 1+\cos 2j\,, \\
2\cos j\,\cos 2j &= \cos j + \cos 3j\,, \\
2\cos j\,\cos 3j &= \cos 2j + \cos 4j\,, \\
2\cos j\,\cos 4j &= \cos 3j + \cos 5j\,, \\
2\cos j\,\cos 5j &= \cos 4j + \cos 6j\,, \\
2\cos j\,\cos 6j &= \cos 5j\,\cos 7j\,, \\
&\text{etc.}
\end{aligned}$$

Now, substituting these for those negative terms, and, at the same time rearranging the terms a bit, we get

$$\begin{aligned}
\cos j - 1 = 2\cos j &+ 2\cos 2j + 2\cos 3j + 2\cos 4j + \text{etc.} \\
-1 \qquad &- \cos j - \cos 2j - \cos 3j - \cos 4j - \text{etc.} \\
-\cos 2j &- \cos 3j - \cos 4j - \text{etc.}
\end{aligned}$$

Note how, when we substituted $1+\cos 2j$ for $2\cos^2 j$, we put the 1 in the second row of the new expression, and the $\cos 2j$ in the third row. Likewise for all the other substitutions. As modern mathematicians, we benefit from the work of Cauchy and we know that such rearrangements of terms may not be valid unless the series involved are absolutely convergent, and that the series in question here are not absolutely convergent. Today, Euler would have to find another way to do this.

Getting back to our formulas, let's rewrite the preceding formula, aligned a bit differently, so that things that cancel can be seen more clearly. We get

$$\begin{aligned}
\cos j - 1 = \qquad\qquad & 2\cos j + 2\cos 2j + 2\cos 3j + 2\cos 4j + \text{etc.} \\
-1 - \cos j \quad & - \cos 2j - \cos 3j - \cos 4j - \text{etc.} \\
& - \cos 2j - \cos 3j - \cos 4j - \text{etc.}
\end{aligned}$$

which is clearly true.

This justifies Euler's claim that, for infinite values of n,

$$t = -\frac{1}{2}.$$

Euler thought he was finished, but the Editor's summary at the beginning of the volume of the *Novi commentarii* mentions that he later added an appendix "Summatio generalis infinitarum aliarum progressionum ad hoc genus referendarum" (Summation of infinitely many general progressions related to this kind). It contains a theorem and two examples.

Theorem: If we know the sum of a progression

$$Az + Bz^2 + Cz^3 + Dz^4 + \mathrm{L} \ + Nz^n,$$

then it always permits us to sum the two progressions

$$S = Ax \sin j \ + Bx^2 \sin 2j \ + Cx^3 \sin 3j \ + \mathrm{L} \ + Nx^n \sin nj$$

and

$$T = Ax \cos j \ + Bx^2 \cos 2j \ + Cx^3 \cos 3j \ + \mathrm{L} \ + Nx^n \cos nj \ .$$

The proof is straightforward, but in the course of the proof, Euler introduces the function notation as he generally uses it in the 1760s. When he writes

$$\Delta : z,$$

he means us to substitute the function defined by the progression

$$Az + Bz^2 + Cz^3 + Dz^4 + \mathrm{L} \ + Nz^n.$$

Though he had used the modern $f(x)$ function notation briefly in the 1730s, Euler did not stick with that notation, and from the 1760s until his death in 1783, he and his assistants used this notation with a symbol, usually an upper-case Greek letter, followed by a colon, and then the variable.

Then he notes that, with p and q as before, that is,

$$p = \cos j \ + \sqrt{-1} \sin j \quad \text{and}$$
$$q = \cos j \ - \sqrt{-1} \sin j \,,$$

his function notation gives

$$2S\sqrt{-1} = \Delta : px - \Delta qx \quad \text{and}$$
$$2T = \Delta : px + \Delta : qx.$$

Then he gives examples.

Example 1: If all the coefficients in ?:z are equal to 1 and if the series is taken to be an infinite series, then

$$\Delta : z = \frac{z}{1-z}.$$

Then, from the equations in the proof of his theorem as well as the identities $p - q = 2\sqrt{-1} \sin j$, $p + q = 2\cos j$ and $pq = 1$, Euler gets that

$$S = \frac{x\sin\varphi}{1-2x\cos\varphi + x^2} \quad \text{and}$$
$$T = \frac{x\cos\varphi - x^2}{1-2x\cos\varphi + x^2}.$$

Typically, Euler checks that his result agrees with what he already knows. In a corollary, he finds that for the special case $x = 1$, this gives back the formulas from earlier in the paper, that

$$S = \frac{\sin\varphi}{2(1-\cos\varphi)} \quad \text{and}$$
$$T = -\frac{1}{2}.$$

As a second example, Euler takes

$$\begin{aligned}\Delta : z &= \ln\frac{1}{1-z}\\ &= z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \cdots,\end{aligned}$$

and he finds that

$$S = \arctan\frac{x\sin\varphi}{1-x\cos\varphi} \quad \text{and}$$
$$T = -\frac{1}{2}\ln\left(1-2x\cos\varphi + x^2\right).$$

We leave the details to the reader.

References:

[E447] Euler, Leonhard, Summatio progressionum $\sin\varphi^{\lambda} + \sin 2\varphi^{\lambda} + \sin 3\varphi^{\lambda} + \cdots + \sin n\varphi^{\lambda}$, $\cos\varphi^{\lambda} + \cos 2\varphi^{\lambda} + \cos 3\varphi^{\lambda} + \cdots + \cos n\varphi^{\lambda}$, *Novi commentarii academiae scientiarum imperialis Petropolitanae*, **18** (1773) 1774, pp. 8-11, 24-36. Reprinted in *Opera omnia* I.15, pp. 168-184. Available online at EulerArchive.org

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