How Euler Did It

by Ed Sandifer

Rational trigonometry

March 2008

Triangles are one of the most basic objects in mathematics. We have been studying them for thousands of years, and the study of triangles, Trigonometry, is, to some extent, a part of every mathematical curriculum. Our oldest named theorem, the Pythagorean theorem, is about triangles, though the theorem was known long before Pythagoras. It is probably our most famous and most often proved theorem as well. Hundreds of different proofs are known, [Loomis 1940] and good writers still find interesting things to say about the theorem. [Maor 2007]

The particular branch of trigonometry where we ask that certain parts of the given triangle, sides, angles, medians, area, etc., is called *rational trigonometry*. Though it originally arose from geometry, rational trigonometry is now usually classified as a part of number theory.

For example, for many people, the Pythagorean theorem is particularly interesting when we consider it as a problem in rational trigonometry and ask that the lengths of the sides of the triangle be whole numbers. This is the problem of so-called Pythagorean triples, three whole numbers a, b and c satisfying

$$a^2 + b^2 = c^2 .$$

As we all know, the simplest such triple is (3,4,5). It is easy to show that there are infinitely many such triples. We can generate all we want by picking two positive integers, m and n, with $m>n$ and letting

$$\begin{aligned} a &= 2mn \\ b &= m^2 - n^2 \\ c &= m^2 + n^2 . \end{aligned}$$

It is easy to check that for these values, indeed, $a^2 + b^2 = c^2$. It is slightly less easy to check that if m and n are relatively prime, one odd and the other even, then a, b and c are pairwise relatively prime, so the method is not just generating infinitely many triangles similar to each other. All Pythagorean triples can be generated in this way.

Another way to generate Pythagorean triples is apparently due to Ozanam. He tells us to look at the sequence of rational numbers

$$1\tfrac{1}{3},\ 2\tfrac{2}{5},\ 3\tfrac{3}{7},\ \ldots, n\tfrac{n}{2n+1}, \ldots.$$

Each of these numbers, written as an improper fraction, $\frac{a}{b}$, gives two of the three numbers of a Pythagorean triple. We leave it to the reader to find why this is true.

Fibonacci also showed a way to find infinitely many different Pythagorean triples, but neither Fibonacci's nor Ozanam's method gives *all* of them.

It should be no surprise that Euler also worked in rational trigonometry. He wrote about half a dozen papers on the subject, and our purpose in this column is to look at a sequence of four of them, giving better and better solutions to the same problem. The first of those papers [E451] gives the problem right in its title, *Solutio problematis de inveniendo triangulo, in quo rectae ex singulis angulis latera opposita bisecantes sint rationales*, "Solution of the problem of finding a triangle in which the lengths of the straight lines drawn from each angle and bisecting the opposite sides are rational." Euler neglects to mention that he means the sides of the triangle to be rational as well, nor that he means to multiply by the least common denominator and make all these measures integers instead of rational numbers. Euler wrote this paper in 1773.

The other three papers, with their titles in English and the years that Euler wrote them, are

E713 (1778) Investigation of a triangle in which the distance from the angles to its center of gravity is rationally expressed
E732 (1779) An easier solution to the Diophontine problem about triangles, in which the straight lines from the angles to the midpoints of the opposite sides are rationally expressed
E754 (1782) A problem in geometry solved by Diophantine analysis

The last of these was written in French. The others were in Latin, though the second one, E713, has a short summary in French, which we quote below:

> This article, which will give pleasure to the small number of amateurs in indeterminate analysis, contains a very beautiful solution to the problem stated in the title. Here it is in just a few words. Let the sides of the desired triangle be $2a$, $2b$, $2c$, and let the straight lines be drawn from their midpoints to the opposite angles, respectively *f, g, h*. Take as you please any two numbers q and r and find $M=\frac{5qq-rr}{4qq}$ and $N=\frac{5rr-9qq}{4rr}$. Reduce the fraction $\frac{(M-N)^2-4}{4(M+N)}$ to its lowest terms, and name the numerator x and the denominator y. Then you will have the side $2a=2qx+(M-N)qy$ and the line $f=rx-\frac{1}{2}(M-N)ry$. Make $p=x+y$ and $s=x-y$, and you will have the sides $2b=pr-qs$ and $2c=pr+qs$ and the lines $g=\frac{3pq+rs}{2}$ and $h=\frac{3pq-rs}{2}$.

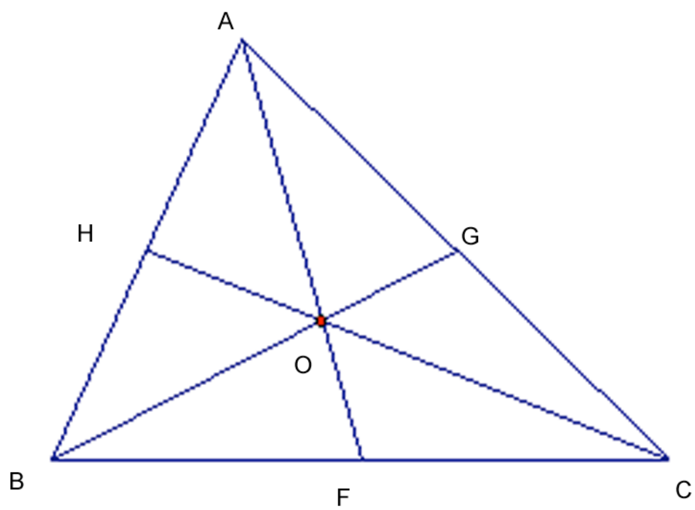
The summary shows that the spirit of Euler's solution is like that of the formulas above that give all the Pythagorean triples. We get to choose two numbers, here q and r, with a few restrictions (like we don't want $M + N = 0$, as stated in the text but not the summary.) Then the formulas give the solutions in terms of p and q. As with the solution to the problem of the Pythagorean triples, it is easy to see that all the values, a, b, c, f, g and h, are indeed rational. It is a bit more subtle and a good deal more tedious to check that these values f, g and h are the medians of the triangle with sides $2a$, $2b$ and $2c$. Some of that will be evident from what follows.

Note that Euler mentions that this paper "will give pleasure to the small number of amateurs in indeterminate analysis." To Euler, "indeterminate analysis" is the practice of finding integer or rational solutions to algebraic equations, what we now call and Euler himself would later call Diophantine analysis. He also mentions that he doesn't think that very many people will be interested, that there are only a "small number of amateurs." I think he uses the word "amateurs" a bit differently than we use the same word today. Now it means "people who are not professionals," but to Euler it meant "people who love the subject." I hope we're all "amateurs" in Euler's sense of the word.

We've seen Euler's beginning, the statement of the problem, and one of his answers. Let's look a bit at his solutions, at some of the things he discovered along the way, and at why he felt the need to return to the problem to improve his solution.

Euler begins the first of his papers, E451 with only the title as preamble and tells us that we should let the sides of the desired triangle be $2a$, $2b$ and $2c$ and the lengths of the medians be f, g and h. Then we want to find rational solutions to the system of equations

$$2bb + 2cc - aa = ff$$
$$2cc + 2aa - bb = gg$$
$$2aa + 2bb - cc = hh$$



Euler calls these three equations his "fundamental equations" for this problem.

He doesn't tell us here why these equations have anything to do with the problem, but in the second of the four papers, E713, perhaps he is being a bit more gentle on his "amateurs," for he gives us details and a diagram.

Let ABC be a triangle, with midpoints F, G and H opposite A, B and C, respectively, and medians AF, BG, and CH intersecting at O, the center of gravity. Let $a=BF=CF$, $b=CG=AG$, $c=AH=BH$, $f=AF$, $g=BG$, $h=CH$ and $\omega=\angle AFB$.

Euler claims, without explicitly mentioning the Law of Cosines, that

$$AB^2 = AF^2 + BF^2 - 2AF \cdot BF \cos\omega$$

and

$$AC^2 = AF^2 + CF^2 + 2AF \cdot CF \cos\omega.$$

Add these to get

$$AB^2 + AC^2 = 2AF^2 + 2BF^2$$

or

$$4cc + 4bb = 2ff + 2aa,$$

or

$$ff = 2cc + 2bb - aa,.$$

Similarly,

$$gg = 2aa + 2cc - bb \text{ and } hh = 2aa + 2bb - cc.$$

Thus the problem becomes to find three numbers, a, b, c, for which these three formulae produce squares.

We'll return to E451 and follow Euler on a short tangent. If we use the three fundamental equations, we find that

$$\begin{aligned} 2gg + 2hh - ff &= 9aa, \\ 2hh + 2ff - gg &= 9bb, \\ 2ff + 2gg - hh &= 9cc. \end{aligned}$$

In the fourth of these papers, E754, Euler describes these equations as "a pleasant property," but that this property "does not contribute in any manner to the solution of the problem." But what is "pleasant" about these equations. They are the same as his three fundamental equations, but with f, g and h substituted for a, b and c, and with $3a$, $3b$ and $3c$ substituted for f, g and h.

This means that if a triangle with sides $2a$, $2b$ and $2c$ has medians of length f, g and h, then a triangle with sides $2f$, $2g$ and $2h$ has medians of length $3a$, $3b$ and $3c$. If the measures in one triangle are all rational, then so are the measures in the other, and so we learn that solutions to this problem in rational trigonometry come in pairs.

But we still don't have any solutions. All of Euler's solutions are rather long, so we will only summarize them

In his first solution, the one given in E451, Euler rewrites his first two fundamental equations as

$$\begin{aligned} ff &= (b-c)^2 + (b-c)^2 - aa = (b-c)^2 + (b+c+a)(b+c-a) \\ gg &= (a-c)^2 + (a+c)^2 - bb = (a-c)^2 + (a+c+b)(a+c-b). \end{aligned}$$

Being a genius at substitution, Euler introduces two new variables, p and q, that enable him to take square roots of these two equations and write them as

$$\begin{aligned} f &= b - c + (b+c+a)p \\ g &= a - c + (a+c+b)q. \end{aligned}$$

After two pages of dense calculations, Euler finds a sixth degree polynomial that gives hh in terms of p and q. Then he finds rational expressions for a, b and c ˆin terms of p and q. Hence, if p and q are rational, then so are a, b, c, f and g. That leaves h. So, all Euler has to do is find some rational values of p an q that make his sixth degree polynomial into a perfect square, and at the same time, don't make any of the denominators of his rational expressions equal to zero. It is tedious, but he manages to find several solutions, among which are

1. $a = 158$ $b = 127$ $c = 131$ $f = 204$ $g = 261$ $h = 255$

and its companion solution, reduced to lowest terms because f, g and h are all multiples of 3,

2. $a = 68$ $b = 87$ $c = 85$ $f = 158$ $g = 127$ $h = 131$.

Five years later, in 1778, Euler made his second attack on the problem, E713. As we mentiond above, the Summary of this article mentions the "Amateurs of analysis," and he uses the law of cosines to justify his fundamental equations.

During those five years, Euler apparently realized that the "pleasant property" was not just an interesting property of triangles with rational medians, but is a general property of all triangles. He calls it a "most distinguished property" and states it more geometrically than he did before, writing

$$AO^2 + BO^2 + CO^2 = \frac{1}{3}\left(BC^2 + AC^2 + AB^2\right).$$

His other calculations are quite similar, but when it comes time to introduce the new variables p and q, he defines them as

$$f = b + c + \frac{p}{q}(b - c + a).$$

This, combined with the first fundamental equation, allows Euler to write a and f, and hence g and h, in terms of b, c, p and q. Things get complicated, but after a while he introduces two more variables, r and s, to make $c + b = pr$ and $c - b = qs$, and then two more, x and y such that $p = x + y$ and $q = x - y$, then t and u so that $\frac{a}{q} = x + ty$ and $\frac{f}{r} = x + uy$, and finally M and N so that $2tx + tty = y + 2Mx$ and $2ux + uuy = y + 2Nx$. In this tower of substitutions, everything ends up depending on q and r, and Euler can find some triangles. We've skipped five pages of details here. The interested reader is encouraged to consult the original sources. The mathematics there is considerably more difficult than the Latin.

In the end, Euler finds that for $q = 1$ and $r = 2$, as well as for $q = 2$, $r = 3$, he gets the same triangle we labeled 1 above, but for $q = 2$, $r = 1$, he finds

3. $a = 404$ $b = 377$ $c = 619$ $f = 942$ $g = 975$ $h = 477$.

Then for $q = 1$ and $r = 3$, he gets

4. $a = 3$ $b = 1$ $c = 2$ $f = 1$ $g = 5$ $h = 4$.

Though this is a solution to the Diophantine equations, the sides 3, 1 and 2 do not form a triangle. He gives several other solutions as well.

Hence, this solution lacks two of the properties we admire in the solution to the problem of Pythagorean triples. Two different choices of the variables p and q can give the same solution, and some choices of p and q can give inadmissible solutions. Euler doesn't seem to ask whether or not all rational triangles with rational medians are generated in this way.

Euler's third solution to the problem of rational medinas, E732, followed just a year later, in 1779. For this paper, he called the midpoints of the sides *X, Y,* and *Z* instead of *F, G* and *H*, and the corresponding lengths of the medians are *x, y* and *z* instead of *f, g* and *h*. Perhaps this is a symptom of Euler's blindness, as he had been almost entirely blind since unsuccessful cataract surgery in 1773, and he was unable to consult his earlier works on the subject to make his notation consistent.

Using his new notation, Euler transforms two of his three fundamental equations into different forms:

I. $xx - yy = 3(bb - aa)$,
II, $xx + yy = 4cc + aa + bb$, and
III. $zz = 2aa + 2bb - cc$.

These equations are enough different from the others that after Euler makes another sequence of miraculous substitutions, introducing *f* and *g*, *p* and *q, m* and *n, t* and *u*, and finally *M*, he gets everything in terms of *f* and *g*. This takes him just three pages of calculations, and the solution is essentially the same as the one we translated above from E754. A few highlights are

$$m = \frac{5gg - ff}{4gg} \text{ and } n = \frac{5ff - gg}{4ff},$$

exactly as *M* and *N* will depend on *r* and *q* in E754. Likewise,

$$p = 4(m+n) \text{ and } q = (m-n)^2 - 4,$$

almost like his variables *x* and *y* are defined in E754, but there he factors out their greatest common divisor.

Now, in terms of *f, g, p* and *q*, Euler tells us that the sides of the triangle are 2*a*, 2*b* and 2*c*, where *a, b* and *c* are given by

$$\begin{aligned} a &= (f-g)p + (f+g)q \\ b &= (f+g)p + (f-g)q \\ c &= 2g(m-n)(3m+n) - 8g \\ &= g(m-n)p + 2gq. \end{aligned}$$

He also gives equations for the lengths of the medians, *x, y* and *z*.

For his first example, he takes $f = 2$, $g = 1$ to get his first solution again, then $f = 1$, $g = 2$ to get the third one. There are no new rational triangles in this paper. Its main improvement over its predecessor, E713, seems to be that its calculations are a bit shorter, and its answer is more concise.

The last of the four papers is much like the third one. Euler wrote it three years later, in 1782, just a year before he died, and for some unknown reason he wrote it in French. The substitutions are slightly different and the resulting algorithm is a bit more streamlined. Moreover, he takes less care to get *integer* results. He is happy to get rational results, then multiply through by a common denominator to make them integer. He gets yet again his examples 1 and 2 above, but this time he gives some new examples, including

5. $a = 159$ $b = 325$ $c = 314$ $x = 309.5$ $y = 188.5$ $z = 202$

From this series of papers, we see that even near the end of his life, Euler went back over his earlier results and tried to improve them. His blindness did not impair his amazing powers of calculation or his ability to design ingenious substitutions. Moreover, while his students mostly worked on applied problems, Euler seemed happy to work also on whimsical problems like this, just because they were fun.

References:

[L] Loomis, Elisha S., *The Pythagorean Proposition: Its Demonstrations Analyzed and Classified, And Bibliography of Sources For Data of The Four Kinds of "Proofs",* 2nd ed., Edwards Brothers, Ann Arbor, MI, 1940. (371 different proofs of the Pythagorean theorem. First edition had 230.)

[M] Maor, Eli, *The Pythagorean Theorem: A 4,000-Year History*, Princeton University Press, 2007.

[E451] Solutio problematis de inveniendo triangulo, in quo rectae ex singulis angulis latera opposita bisecantes sint rationales, *Novi commentarii academiae scientiarum Petropolitanae* **18** (1773) 1774, pp. 171-184 Reprinted in *Opera omnia* I.3, pp, 282-296. Available at EulerArchive.org.

[E713] Euler, Leonhard, Investigatio trianguli in quo distantiae angulorm ab eius centro gravitates rationaliter exprimantur, *Nova acta academiae scientiarum Petropolitanae,* **12** (1794) 1801 pp 68-69, 101-113. Reprinted in *Opera omnia* I.4 pp. 290-302. Available at EulerArchive.org.

[E732] Euler, Leonhard, Solutio facilior problematis Diophantei circa triangulum, in quo rectae ex angulis latera opposita bisecantes rationaliter exprimantur, *Mémoires de l'académie des sciences de St.-Pétersbourg*, **2** (1807/1808) 1810, pp. 10-16. Reprinted in *Opera omnia* I.4, pp. 399-405. Available at EulerArchive.org.

[E754] Euler, Leonhard, Problème de géométrie résolu par l'analyse de Diophante, *Mémoires de l'académie des sciences de St-Pétersbourg* **7** (1815/16), 1820, pp. 3-9. *Opera omnia* series I volume 5 pp. 28-34. . Available at EulerArchive.org.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 35 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org). His first book, *The Early Mathematics of Leonhard Euler*, was published by the MAA in December 2006, as part of the celebrations of Euler's tercentennial in 2007. The MAA published a collection of forty *How Euler Did It* columns in June 2007.
