



## Gamma the constant

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Sam Kutler, now retired from St. John’s College in Annapolis, once pointed out that there are three great constants in mathematics,  $\pi$ ,  $e$  and  $\gamma$ , and that Euler had a role in all three of them. Euler did not discover  $e$  or  $\pi$ , but he gave both of them their names. In contrast, Euler discovered, but did not name  $\gamma$ , the third and least known of these constants.

This  $\gamma$  is usually known as the Euler-Mascheroni constant, acknowledging both the work Euler did in discovering the constant in about 1734, (more on this later) and the work of Lorenzo Mascheroni (1750-1800). Mascheroni was a priest and a professor of mathematics at the University of Pavia in Italy. As Mascheroni studied Euler’s books on integral calculus, he took careful notes and extended several of Euler’s results, especially those involving the constant that now bears his name. Mascheroni published his notes in 1790 under the title *Adnotationes ad calculum integrale Euleri*. The editors of Euler’s *Opera Omnia* have republished Mascheroni’s *Adnotationes* as an appendix to the second volume of Euler’s integral calculus in Series 1 volume 12 of the *Opera Omnia*. Mascheroni’s *Adnotationes* are a model of a wonderful way to learn mathematics: find an excellent book on the subject and work through it, theorem by theorem, working examples, checking proofs, and extending results when you can. It doesn’t make very exciting reading, though. It’s a bit like reading someone else’s homework assignments, watching that person struggle, but eventually master difficult concepts.

Euler made his first steps towards discovering gamma the constant in the same letter to Christian Goldbach dated October 13, 1729 in which he also first mentioned gamma the function. Goldbach and Daniel Bernoulli had been working on “interpolating a sequence” by finding a function that “naturally expresses” the sequence, and that is also defined for fractional values. We learned in last month’s column how the gamma function interpolates the series of factorial numbers.

Bernoulli and Goldbach were also working, without much success, to interpolate the partial sums of the harmonic series. In modern notation, they were looking for a function  $f(x)$  such that, if  $n$  is a positive integer, then  $f(n) = \sum_{k=1}^n \frac{1}{k}$ . In the letter, Euler hinted at his solution, claiming that he had found such a function, and that  $f\left(\frac{1}{2}\right) = 2 - 21n2$ , but Euler did not give details. Instead, he published the details a few years later as *De summatione innumerabilium progressionum*, “On the summation of innumerably many progressions.” [E20]

In his article, Euler asks us to look at the integral  $\int_0^1 \frac{1-x^n}{1-x} dx$ . If  $n$  is a positive integer, we can expand the integrand as a geometric series and get

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \dots + x^{n-1}.$$

Integrating this gives

$$\begin{aligned} \int_0^1 \frac{1-x^n}{1-x} dx &= \int_0^1 (1 + x + x^2 + \dots + x^{n-1}) dx \\ &= \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} \right) \Big|_{x=0}^{x=1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \end{aligned}$$

Taking  $f(n) = \int_0^1 \frac{1-x^n}{1-x} dx$  gives the  $n$ th partial sum of the harmonic series. Since the integral is well-defined even if  $n$  is not an integer, we see that  $f$  is a function that interpolates the partial sums of the harmonic series. It is the function that Bernoulli and Goldbach had been unable to discover.

If we take  $n = 1/2$ , we get, by a rather tricky bit of integration, the details of which we will omit, just like Euler did,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \int_0^1 \frac{1-\sqrt{x}}{1-x} dx \\ &= \int_0^1 \frac{1}{1-\sqrt{x}} dx \\ &= \left( 2\sqrt{x} - 2\ln(1+\sqrt{x}) \right) \Big|_{x=0}^{x=1} \\ &= 2 - 2\ln 2 \end{aligned}$$

as Euler had claimed in his letter to Goldbach.

Euler tries a similar trick on the sum of the reciprocals of perfect, taking a stab at the Basel problem, but he gets an integral that he's unable to evaluate. He does manage to approximate it, though, and thus makes a major step towards his solution to the Basel problem, which he discovered just two years later. Details are in my book. [S]

In both of these projects, partial sums of the harmonic series, and the Basel problem, Euler notices how sums like  $\sum_{k=1}^n f(k)$  are closely related to integrals  $\int_0^n f(x) dx$ , and that as  $n$  gets large, the difference between the two seems to converge to a constant.

In E20, Euler didn't follow up on this idea, but he soon returned to it with E25, *Methodus generalis summandi progressionis*, "General methods of summing progressions." There he gives his first account of what we now call Euler-Maclaurin summation. Using Euler's notation, if  $s$  is the sum of a sequence the terms of which are "naturally expressed" by a function  $t(n)$ , then Euler found that

$$s = \int t \, dn + at + \frac{bdt}{dn} + \frac{gd^2t}{dn^2} + \frac{dd^3t}{dn^3} + \text{etc.}$$

where the Greek letters are constants that will eventually turn out to be related to the Bernoulli numbers. It will take Euler 20 years to discover that relationship, though.

This formula tells us that the difference between a sum,  $s$ , and an integral,  $\int t \, dn$ , is equal to  $at + \frac{bdt}{dn} + \frac{gd^2t}{dn^2} + \frac{dd^3t}{dn^3} + \text{etc.}$  For some functions  $t$ , this error term converges quickly and is easy to estimate. Maclaurin was interested in using finite sums to approximate definite integrals, but Euler used improper integrals to approximate infinite series and definite integrals to approximate finite sums. Since the Euler's and Maclaurin's approaches were so different, the summation formula bears both their names, even though Euler found his version at least eight years before Maclaurin's work.

For Euler and Maclaurin, each series or function had its own error term, given by the formula above. They were able to calculate the error terms for particular cases. The error term for  $1/x$  is approximately 0.577, the value we now call  $\gamma$ , while for  $1/x^2$  it is approximately 0.645. Mascheroni put his own name next to Euler's in 1790 by calculating the error terms corresponding to many other series and functions, and showing how all the error terms involved the value  $\gamma$ .

Finally we turn to the question, who first called it  $\gamma$ ? Havil [H, p. 90]. Dunham [D] and Glaisher [G] all tell us that it was Mascheroni. Twice I have checked the *Opera Omnia* edition of Mascheroni's *Adnotationes*, and Mascheroni consistently uses the symbol  $A$ . Jeff Miller [Mi] cites a source that says it was Euler in 1781 who first used  $\gamma$ . I have checked all of Euler's 1781 works, and I find him using  $A$  and  $C$ , but not  $\gamma$ .

260 15. *Bretschneider, theoriae logarithmi integralis lineamenta nova.*

$$7. \quad \ln(1+x) = c + \int \frac{[(1+x)]^0}{1!} \cdot \frac{\partial x}{x} + \int \frac{[(1+x)]^1}{2!} \cdot \frac{\partial x}{x} + \int \frac{[(1+x)]^2}{3!} \cdot \frac{\partial x}{x} + \dots$$

Jam vero a cl. *Eulero* est demonstratum, esse

$$8. \quad (-1)^n \int_0^1 \frac{[(1-x)]^n}{n!} \cdot \frac{\partial x}{x} = \frac{\sigma_0}{n \cdot n!} + \frac{\sigma_1}{(n+1)(n+1)!} + \frac{\sigma_2}{(n+2)(n+2)!} + \dots$$

$$= \sigma_{n+1} = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \dots$$

Adhibitis igitur in aequatione (7.) signis inferioribus et posito  $x=1$ , invenitur

$$9. \quad c = \frac{1}{2}\sigma_1 - \frac{1}{3}\sigma_2 + \frac{1}{4}\sigma_3 - \frac{1}{5}\sigma_4 + \dots,$$

qui est valor quantitatis illius constantis, quam vir cl. *Kramp* in summatione seriei harmonicae investigavit et in numeros sequentes  $\gamma, = c = 0,577215\ 664901\ 532860\ 618112\ 090082\ 3..$  computavit.

Miller and Glaisher also cite an 1835 article by Carl Anton Bretschneider [B], and, indeed, in the paragraph shown above, from page 260 of that article, we find Bretschneider using  $\gamma$  to denote the constant. We also see him citing Euler and, elsewhere on the page, Mascheroni, and the work of “vir cl. *Kramp*,” who calculated  $\gamma$  to 31 decimal places. This is curious, since the value cited is the same as the one calculated by Mascheroni, given to 31 decimal places but correct only to 19 places. The value to 31 places generated by Maple™ is 0.577215 664901 532860 6.

So, who *really* first called it  $\gamma$ ? As is the case for many questions of the form “who was the first to ...?” I’m not sure, but I don’t think it was Euler or Mascheroni. Though the history of the constant  $\gamma$  is confusing and riddled with errors, and the secondary sources disagree, [D, G, H, Mi] I think it was probably Bretschneider. He’s not very famous, and perhaps he deserves to be known for this, if for nothing else.

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