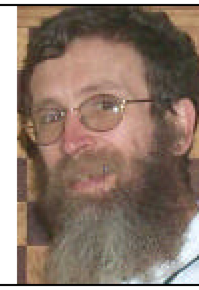
How Euler Did It

by Ed Sandifer



e, π and i: Why is “Euler” in the Euler identity?

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One of the most famous formulas in mathematics, indeed in all of science is commonly written in two different ways:

$$e^{p i} = -1 \text{ or } e^{p i} + 1 = 0 .$$

Moreover, it is variously known as the Euler identity (the name we will use in this column), the Euler formula or the Euler equation. Whatever its name or form, it consistently appears at or near the top of lists of people’s “favorite” results. It finished first in a 1988 survey by David Wells for *Mathematics Intelligencer* of “most beautiful theorems.” It finished second in a 2004 survey by the editors of *Physics World* to select the “greatest equations” and it was third in a 2007 survey of participants in an MAA Short Course of “Euler’s greatest theorems.”[1]

Whether people call it a formula, an equation or an identity, and regardless of which form they use, almost everyone credits the result to Euler. But it is not entirely clear *why* people give him credit for this result, because he never wrote it down in anything remotely like this form, because he wasn’t the first one to know the fact behind the formula, and because he himself credited that fact to his mentor, Johann Bernoulli. In this column we will look at the origins of the Euler identity, see what Euler contributed, and consider whether it is correctly named.

Phase 1: 1702 to 1729

There are two formulas that are closely related to the Euler identity. The first we will call the “Euler formula”:[2]

$$e^{iq} = \cos q + i \sin q$$

The Euler identity is an easy consequence of the Euler formula, taking $q = p$. The second closely related formula is DeMoivre’s formula:

$$(\cos q + i \sin q)^n = \cos nq + i \sin q \ .$$

[1] See “Euler’s Greatest Hits”, *How Euler Did It*, February 2006, or pages 1-5 of your columnist’s new book, *How Euler Did It*, a collection of 40 of these columns published just last month by the Mathematical Association of America.

[2] Throughout this column, we will use *i* to denote $\sqrt{-1}$, even though Euler did not introduce the more convenient *i* notation until the 1770’s, long after the events in this story.

This, too, is an easy consequence of the Euler formula, since

$$\left(\cos q + i\sin q\right)^n = \left(e^{iq}\right)^n = e^{inq} = \cos nq + i\sin nq \ .$$

The relation between DeMoivre's formula and the Euler identity will turn out to be deeper than this.

The English mathematician Roger Cotes (1862-1716) was studying problems in the arc length of spirals. In about 1712, in the course of his investigations, he seems to be the first one to discover a formula equivalent to the Euler formula:

$$\ln\left(\cos q + i\sin q\right) = iq \ .$$

This is easily transformed into the Euler formula by exponentiating both sides, but apparently Cotes never did this. Moreover, Cotes died rather suddenly in 1716 without publishing much of his work on this subject.

I've not found much about Abraham DeMoivre's (1667-1754) discovery of his formula, what he was thinking when he found it, or how rigorous his derivations were. Mactutor [McT] tells us:

> It appears in this form in a paper which de Moivre published in 1722, but a closely related formula had appeared in an earlier paper which de Moivre published in 1707.

Both DeMoivre and Cotes lived in England, though, and in those years of the Newton-Leibniz dispute, Continental mathematicians sometimes made a special effort to ignore English mathematical results.

Meanwhile, on the other side of the English Channel, Johann Bernoulli (1667-1748) was uncovering some of the first geometric properties of complex numbers. In 1702 he gave a formula [Br] for the area of a sector of a circle of radius *a*, centered at the origin between the *x*-axis and the radius to the point (*x*, *y*) as

$$\frac{aa}{4\sqrt{-1}}\ln\frac{x+y\sqrt{-1}}{x-y\sqrt{-1}} \ .$$

Twenty-five years later, in 1727, Bernoulli was studying the equation $y = \left(-1\right)^x$ with his young student Leonhard Euler. In the course of their discussions, they had to figure out the nature of logarithms of negative numbers. Bernoulli had argued that ln(–1) = 0, since

$$0 = \ln(1) = \ln(-1\cdot -1) = 2\ln(-1) \ .$$

The same argument applies to any negative number. They were perplexed because they had equally convincing (and flawed) arguments to "prove" that $\ln\left(-x\right) = \ln\left(x\right)$.

Euler took $x = 0$ in Bernoulli's 1702 formula to find the area of a quarter circle. He reasoned that $\frac{aa}{4\sqrt{-1}}\ln\left(-1\right)$ was finite and non-zero. But if Bernoulli were correct that ln(–1) = 0, the area would be zero. Bernoulli was unconvinced, and the issue faded. More details of this episode are given in [Br].

If Euler had taken this argument just one step farther and noticed that the area of a quarter circle is $\frac{p\,a^2}{4}$, he could have solved the equation $\frac{a^2}{4i}\ln(-1)=\frac{p\,a^2}{4}$ and found that $\ln(-1)=p\;i$. From this it follows immediately that $e^{p\,i}=-1$, but Euler did not take this step.

Two years later, Euler was writing [E19] his pioneering work on the gamma function. In one of his examples, he tells us that a particular infinite product turns out to be [S] "$\frac{1}{2}\sqrt{i\ln(-1)}$, which is equal to the side of the square equal to the circle with diameter 1."

Decoding this is a little tricky. On the one hand, "the side of the square equal to the circle with diameter 1" tells us first to find the area of a circle with diameter 1, that is $\frac{p}{4}$, then to find a square with the same area, and to find the length of the side of that square. This gives $\frac{\sqrt{p}}{2}$. Setting this equal to $\frac{1}{2}\sqrt{i\ln(-1)}$ and applying a bit of algebra, it is easy to conclude that $e^{p\,i}=-1$. Euler doesn't do this, though, nor does he explain his claim that $\frac{1}{2}\sqrt{i\ln(-1)}=\frac{\sqrt{p}}{2}$ or why it is equal to the infinite product he had been studying.

By 1729, we have four different people, DeMoivre, Cotes, Bernoulli and Euler (twice), who have found the essential fact behind the Euler identity, but none of them have recognized its importance or written it in anything like the form we recognize today.

Phase 2: The 1740s

Let's jump forward to the 1740s, when Euler was writing his great precalculus textbook, the *Introductio in analysin infinitorum* [E101]. Euler spent most of the 1740s writing this book, then had trouble finding a publisher. Eventually he found a publisher in Switzerland and the book came out in 1748. Today, many people think it is the greatest mathematics book ever written.

Chapter 8 of Euler's *Introductio* is titled "On transcendental quantities which arise from the circle." It is the first time that anyone treats sines, cosines, etc. as functions rather than as ratios, and so it makes an important step towards making functions a fundamental object in mathematics.

Euler spends the first part of the chapter establishing the basic properties of the sine, cosine and tangent functions, very much the way we do them today. Then he begins using complex numbers. He tells us,[3] "Since $(\sin z)^2+(\cos z)^2=1$, we have the factors $(\cos z+i\sin z)(\cos z-i\sin z)=1$."

He then asks us to

> [c]onsider the following product: $(\cos z+i\sin z)(\cos y+i\sin y)$, which results in
>
> $\cos y\cos z-\sin y\sin z+(\cos y\sin z+\sin y\cos z)i$, which results in …
>
> $(\cos y+i\sin y)(\cos z+i\sin z)=\cos(y+z)+i\sin(y+z)$

[3] We use the Blanton translation, published by Springer in 1988 and 1990.

Since multiplication can be regarded as repeated addition, a few lines later he shows that

$$(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz \ .$$

This, of course, is DeMoivre's formula. It is not clear whether or not Euler knew of DeMoivre's work, but in the *Introductio* he does not usually cite sources. He also does not seem to consider the possibility that this formula might be true even if n is not an integer.

From DeMoivre's formula he calculates that

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}, \text{ and}$$
$$\sin nz = \frac{(\cos z + i \sin z)^n - (\cos z - i \sin z)^n}{2i}$$

He boldly takes z to be infinitely small, so that $\sin z = z$ and $\cos z = 1$, and then takes n to be an infinitely large number with $nz = n$, where n is finite, and gets the Taylor series for sine and cosine. Readers who are anxious about the 18th-century use of infinite and infinitesimal numbers may either read the first four chapters of the *Introductio* to become more familiar with the practice, or they can re-cast Euler's argument into the language of limits. Either way, it is very beautiful mathematics.

A few paragraphs later he uses this version of DeMoivre's formula, taking z infinitely small, j infinitely large and $jz = n$, where again n is finite, to get

$$\cos n = \frac{\left(1 + \frac{in}{j}\right)^j + \left(1 - \frac{in}{j}\right)^j}{2} \text{ and}$$
$$\sin n = \frac{\left(1 + \frac{in}{j}\right)^j - \left(1 - \frac{in}{j}\right)^j}{2i}.$$

But when j is an infinite number, $e^z = \left(1 + \frac{z}{j}\right)^j$ so these formulas are equivalent to

$$\cos n = \frac{e^{in} + e^{-iv}}{2} \text{ and } \sin n = \frac{e^{in} - e^{-in}}{2i}.$$

Now comes the *coup de grace*. Multiply these equations by 2 and $2i$, respectively, and add them together to get the Euler formula:

$$e^{in} = \cos n + i \sin n \ .$$

Euler moves on to apply these results to the practical problems of calculating sines and cosines, without ever considering the special case $n = p$ and without explicitly writing down the Euler identity.

The Judgment of History

Early in the 1700s, Cotes, DeMoivre, Johann Bernoulli and Euler himself all had the pieces that could have led them to discover the Euler formula. The problems they were working on did not depend on the Euler formula, though, so none of them had any reason to discover the formula at the time.

In contrast, in the 1740s Euler had good reasons to know the Euler formula, discovering properties of trigonometric functions and finding good ways to approximate them. Moreover, he had a beautiful and convincing demonstration of the Euler formula, satisfying all the standards of rigor of the time and easily translatable into the modern language of limits. Since Euler's presentation was both complete and well-motivated, it seems like the right thing to do to attach his name to the formula.

The name of the Euler *identity* presents a slightly different problem. Though it is only a special case of the Euler formula, it seems that he never wrote it down. I have made no progress in finding who was the first to do so. The mathematical community seems content and almost unanimous in calling it the Euler identity, and nobody else seems to have a claim that is nearly as good.

And it is one of the most beautiful formulas in all of mathematics.

References:

[Br] Bradley, Robert E., "Euler, D'Alembert and the Logarithm Function", in *Leonhard Euler: Life, Work and Legacy*, Bradley, R. E. and C. E. Sandifer, editors, pp. 255-277, Elsevier, Amsterdam, 2007.

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[E101] Euler, Leohnard, *Introductio in analysin infinitorum*, Bosquet, Lausanne, 1748. Available at EulerArchive.org. English translation by John Blanton, Springer, New York, 1988 and 1990.

[McT] O'Connor, J. J. and E. F. Robinson, "DeMoivre", *The MacTutor History of Mathematics Archive,* http://www-groups.dcs.st-and.ac.uk/~history/Biographies/De_Moivre.html.

[S] Sandifer, C. Edward, *The Early Mathematics of Leonhard Euler*, Mathematical Association of America, Washington, DC, 2007.

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How Euler Did It is updated each month.