





by Ed Sandifer

St. Petersburg Paradox

July 2007

If we knew what we were doing, it would not be called research, would it? -attributed to Albert Einstein (1879-1955)

When I was first learning probability as an undergraduate, I learned about something called "the St. Petersburg Paradox." One version of this paradox is as follows:[J]

A man is to throw a coin until he throws head. If he throws head at the *n*th throw, *and not before*, he is to receive $\pounds 2^n$. What is the value of his expectation?

We learned about this just after we learned about geometric distributions, so we knew that the probability of throwing *n* heads in a row was $\frac{1}{2^n}$. Our instructor, David Griffeath,¹ asked us to do two things with this problem, to find the average number of times the coin would be tossed in this game, and to find the so-called "value of the game," the amount of money that a player who plays this game should expect to win. We had just learned some formulas for these values, but the infinite series behind those formulas are pretty straightforward. The expected number of tosses is²

$$\sum_{n=1}^{\infty} np(n) = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2,$$

and the value of the game is

$$\sum_{n=1}^{\infty} 2^n p(n) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty.$$

¹ now at the University of Wisconsin, not to be confused, as I did, with the statistician at the University of Wollongong, David Griffiths. I had assumed they were the same person, one of several over-hasty assumptions I made in writing this column.

² A really beautiful proof of this summation is due to an obscure 14th century English mathematician named Richard Swineshead.

At the time, I thought that the "paradox" was that a game with such a short expected duration could still have an infinite value.

Let us move forward several years to an AMS Section meeting in Hoboken, NJ on April 15, 2007, the very day of Euler's 300th birthday. Robert Bradley, President of The Euler Society, gave a talk on Euler's probability and statistics, and he mentioned that one of Euler's posthumously published papers [E811], *Vera aestimatio sortis in ludis*, (On the true valuation of the risk in games) and said that it described the St. Petersburg paradox. I assumed that Euler had posed the St. Petersburg paradox, and resolved to write a column about it.

Euler's article was published in 1862, seventy-nine years after his death in 1783, as part of a two-volume set called the *Opera posthuma*. It is quite short, only four pages in the original, four pages in Pulskamp's English translation, and, because of the extensive footnotes, eight pages in the *Opera omnia*. It was one of the articles that Euler did not bother to see published during his lifetime. Later, we will speculate on why he might have done this.

Moreover, E811 is one of the articles for which Gustav Eneström [En] does not give a date written. Later, we will hazard a guess on this as well.

Turning to the article itself, we see that Euler begins by acknowledging the work of Blaise Pascal, Christiaan Huygens and Jakob Bernoulli. His acknowledgement turns almost immediately to criticism when he writes:³

Following the way of thinking of these men I do not act imprudently if I take up a game where it may happen equally easily that I would either win or lose 100 Rubles. But if all of my wealth is worth only 100 R., I seem to myself to begin this game about to be played most imprudently.

Euler is telling us that to a person with only 100 Rubles, that money is worth more to him than the next 100 Rubles would be. He has discovered one of the properties of what we now call a *utility function*, that the value an individual places on an amount of money depends on how much money that person already has. Usually, the more money you already have, the less value you place on having another Ruble (or dollar), so we would say that utility functions have decreasing first derivatives. This is an important concept in modern economic theory.

He goes on to describe the game behind the St. Petersburg paradox, though he doesn't call it that. Euler does make one trivial change to the game. Rather than tossing a coin, he has us roll a die. For him, the game ends if we roll an odd number, so we win by rolling lots of even numbers.⁴

Now Euler brings a new name into the story, Nicolaus Bernoulli,⁵ nephew of the bickering brothers Jakob and Johann. The former of these brothers wrote *Ars conjectandi*, the first comprehensive study of the theory of probability, published in 1713, and the latter was Euler's mentor in Basel.

³ I follow the Pulskamp translation throughout, unless noted otherwise.

⁴ As I write this column, I've been reading Rudolf Taschner's little book [Ta], in which he tells us that ancient Greek and Hebrew numerologists considered even numbers *un*lucky, and odd ones lucky. Euler's description makes the even ones lucky.

⁵ There were many Bernoullis. This was the one we call Nicolaus I, (1687-1759). Though both his father (1662-1716) and his grandfather (1623-1708) were named Nicolaus, they were not mathematicians, so sources in the history of mathematics do not give them numbers. It is easy to confuse Nicolaus I with his cousin, Nicolaus II (1695-1726), son of Euler's mentor Johann and a childhood friend of Euler himself.

According to Euler, Nicolaus believed that, based on his ideas of people's values, given a choice of playing the St. Petersburg game once or receiving a certain payment of 20 Rubles, most people would choose the 20 Rubles. It is interesting that Nicolaus Bernoulli was not among the authors Euler cited earlier in his article, probably because he found reason to criticize Huygens, *et al.*, but he agreed with Nicolaus. It is also interesting that Nicolaus didn't publish much, and apparently didn't publish anything about the St. Petersburg Paradox.

Euler gives some thought to how a man with 20 Rubles would be reluctant to play a game in which he has equal chances of either winning or losing his 20 Rubles, but a very wealthy person with, worth many Rubles (Euler doesn't tell us how many Rubles it took to be "wealthy.") would not be nearly so reluctant. Following a typical Eulerian analysis, he introduces the idea of a *status*, roughly how much the player's wealth would be worth after a particular outcome. If a game or business deal⁶ has equal chances of bringing a player to status b or to status c, then Euler proposes that the game be worth \sqrt{bc} , that is, the geometric mean of b and c. As a further example, if a game has three equally likely outcomes, leading to statuses b, c or d, then it would be worth $\sqrt[3]{bcd}$.

Probability was a new subject at the time, and analysts hadn't made the step from discrete probability to continuous probability, so Euler's most complicated analysis in this direction has us suppose that there are *m* ways that the player can be rewarded with status *a*, *n* ways to get *b* and *p* ways to arrive at status *c*. We would say that the corresponding probabilities are $\frac{m}{m+n+p}$, $\frac{n}{m+n+p}$ and $\frac{p}{m+n+p}$. In this case, Euler's idea gives the value ${}^{m+n+p}\sqrt{a^m b^n c^p}$. He points out that Huygens would assign the value $\frac{ma+nb+pc}{m+n+p}$ to the same game, while the logarithm of Euler's value would be $\frac{m\ln a + n\ln b + p\ln c}{m+n+p}$.

Next we take our player's current status to be *A*, and give the player even chances of either winning *a* or losing *b*. In this case, where Hugyens' valuation would give the game value of $\frac{a-b}{2}$, Euler's valuations would have us compare $\sqrt{(A+a)(A-b)}$ with *A*. Euler says that he would be indifferent about playing this game if $\sqrt{(A+a)(A-b)} = A$, which happens if $A = \frac{ab}{a-b}$. On the other hand, if a = b, Euler notes that $\sqrt{(A+a)(A-a)} = \sqrt{A^2 - a^2}$ and that is always less than *A*, "unless my resources be infinite." For finite values of *A*, he notes that for such a game he expects to lose

$$A - \sqrt{A^2 - a^2} = \frac{1}{2} \cdot \frac{a^2}{A} + \frac{1}{8} \cdot \frac{a^4}{A^3} + \frac{1}{16} \cdot \frac{a^6}{A^5} + \frac{5}{128} \cdot \frac{a^8}{A^6} + \text{etc.}$$

Then, suddenly, without explaining this series or "solving" the St. Petersburg paradox, the paper ends. It looks like Euler didn't publish this paper in his lifetime because he didn't finish it.

⁶ The introduction of business deals as being subject to the same kind of analysis as games of chance is an exciting innovation.

Now we must ask *why* Euler abandoned this paper. With the evidence available today, we cannot know for sure, so let us speculate.

Though this series expansion seems to be a bit of a dead-end, it is an interesting exercise (left to the reader) to apply Euler's valuation (not considering the current status parameter A) to the St. Petersburg paradox. It gives a nice, reasonable, finite valuation, so it seems unlikely that Euler quit because he couldn't think of anything else to write on the subject. Apparently he set it aside for some other reason.

Let's try to guess when Euler wrote E811. Most of his papers on probability date from his Berlin years, especially in the early 1750s, when Frederick the Great asked him to do some work on the national lottery and on life insurance. He wrote all of those papers in French, though, the language of the Berlin Academy, and this paper is in Latin. To me, this makes it seem more likely that Euler wrote it in St. Petersburg, either as a young man between 1728 and 1741, or in his second St. Petersburg years, 1766 to 1783.

But if he had written E811 sometime after 1766, it seems that Euler would have mentioned some of the work of d'Alembert [d'A] or Daniel Bernoulli⁷ [B] or the correspondence between Nicolaus Bernoulli and Pierre Rémond de Montmort, [M], all mentioned in Todhunter [To].⁸

A clue! Daniel Bernoulli! Didn't I read a footnote somewhere that said Daniel Bernoulli and Leonhard Euler were friends, and that they lived and worked together in St. Petersburg in the 1730s? So I looked for Bernoulli's article and was delighted to find that it had been translated into English, published in *Econometrica* in 1954 and was available on JSTOR. The article, *Specimen theoriae novae de mensura sortis*, (Exposition of a new theory on the measurement of risk) is brilliant and complete. Many of his ideas are equivalent to Euler's, but expressed quite differently. For example, rather than hypothesizing Euler's geometric mean property valuing games, Bernoulli proposes a principle:

... in the absence of the unusual, the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed.

From this, he shows that the utility function is a logarithmic curve and then derives Euler's geometric mean property. He derives the same formula for the minimum wealth an individual must

have before being willing to undertake a particular risky venture, $A = \frac{ab}{a-b}$, though Bernoulli uses

different symbols. Bernoulli, though, in a style that would later be typically Eulerian, does some wellchosen examples involving characters he calls Peter and Paul in one example, then Caius and Sempronius, respectively, in a second and a third example. He uses his utility function to "solve" the St. Petersburg paradox (readers who want to check their work can refer to \$19 of Bernoulli's article), both for an initial fortune of zero and for an arbitrary status that Bernoulli denotes by **a**.

⁷ This is Daniel I (1700-1782), another son of Johann, not Daniel's nephew Daniel II (1751-1834). Daniel I was another childhood friend of Euler in Basel and senior colleague in St. Petersburg. When Euler first arrived in St. Petersburg, he rented a room in Daniel's house.

⁸ Todhunter, writing in 1865, doesn't mention Euler's E811 at all, probably because it had been published only three years earlier, in 1862. Todhunter does describe several of Euler's other contributions to the subject.

As an addendum to the paper, Daniel tells us that he sent a copy of the paper to his cousin, Nicolaus Bernoulli, who admired the results and directed Daniel to related work done by Cramer⁹ in 1728. I've not seen this work of Cramer. Both Daniel and Nicolaus admired it, but it seems that Todhunter did not.

Now, using what we know, let's guess when and why Euler didn't finish E811. Perhaps this is what happened.

Sometime around 1730 or 1731, Daniel Bernoulli learned of the St. Petersburg paradox, perhaps by reading his cousin's correspondence in Montmort's book, or perhaps from Nicolaus himself. He shared the problem with his friend and colleague Leonhard Euler, and they both went to work on the problem. Bernoulli had the advantages of a head start and more access to the thoughts of Montmort and of his cousin Nicolaus. Moreover, Daniel was seven years older and at the time he was probably better than the 23-year old Euler was at these kinds of things. For whatever reason, Bernoulli finished his paper first, and it was better than Euler's. Euler recognized this and set his aside.

The Einstein quote at the beginning of this column was chosen to warn the reader that this story would not proceed in a straight line, from beginning to end. Instead, there are false steps, incorrect assumptions, and things happening out of order. Let us review some of the highlights:

- Euler probably wrote E811 in 1730 or 1731.
- He abandoned it because Daniel Bernoulli's article on the same subject was better.
- The "paradox" in the St. Petersburg problem is not its infinite expected value, but that no reasonable person would pay a large sum to be allowed to play the game.
- David Griffiths, the Wollongong Statistician, is not David Griffeath, probabilist at the University of Wisconsin and chef at the Primordial Soup Kitchen.
- Euler did not solve the St. Petersburg paradox. Euler was not the first person to do everything.

And finally,

If you know when you start how it's going to end, it doesn't make a very good story, does it?
Ed Sandifer (1951 -)

References:

- [B] Bernoulli, Daniel, "Specimen theoriae novae de mensura sortis", Commentarii academiae scientiarum imperialis Petropolitanae, 5 (1731) 1738, pp. 175-192. Translated by Dr. Louise Sommer as "Exposition of a new theory on the management of risk", Econometrica 22 (1) (January 1954), pp. 23-36. Available on JSTOR.
- [d'A] d'Alembert, Jean le Rond, "Doutes et questions sur le calcul des probabilities", *Mélanges de literature et de philosophie* **5** (1767).
- [E811] Euler, Leonhard, "Vera aestimatio sortis in ludis", *Opera posthuma* **1**, pp. 315-318, 1862. Translation by Richard Pulskamp titled "The true valuation of the risk in games" available at http://www.cs.xu.edu/math/Sources/Euler/E811.pdf
- [En] Eneström, Gustav, "Verzeighness der Schriften Leonhard Eulers", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ergänzungsband **4**, Teubner, Leipzig, 1910-1913. English translation by Greta Pearl available at EulerArchive.org.
- [J] Jackson, C. S., "The St. Petersburg Problem", *The Mathematical Gazette*, **8** (March 1915) p. 48. Available on JSTOR.
- [M] Montmort, Pierre Rémond de, Essai d'Analyse sur les Jeux de Hazards, 2ed., Jombert, Paris, 1714.
- [Ta] Taschner, Rudolf, *Numbers at Work: A Cultural Perspective*, A. K. Peters, Wellesley, MA, 2007.

⁹ Swiss mathematician Gabriel Cramer (1704-1752) of "Cramer's Law"

[To] Todhunter, Isaac, *A History of the Mathematical Theory of Probability from the Time of Pascal to That of Laplace*, Macmillan, Cambridge & London, 1865. Reprinted by Chelsea, 1949.

Ed Sandifer (<u>SandiferE@wcsu.edu</u>) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 35 Boston Marathons on his shoes, and he is Secretary of The Euler Society (<u>www.EulerSociety.org</u>). His first book, *The Early Mathematics of Leonhard Euler*, was published by the MAA in December 2006, as part of the celebrations of Euler's tercentennial in 2007. The MAA has just published a collection of the *How Euler Did It* columns in June 2007.

How Euler Did It is updated each month. Copyright ©2007 Ed Sandifer