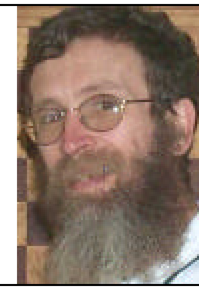
How Euler Did It

by Ed Sandifer



Partial fractions

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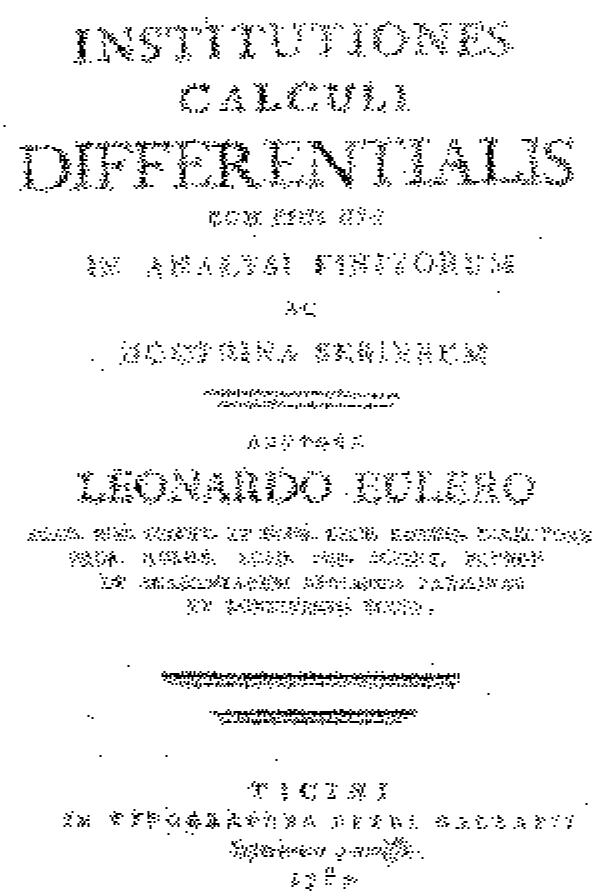
Sometimes Euler has a nice sense of showmanship and a flair for a "big finish." At the end of a long, sometimes difficult work, he'll put a beautiful or particularly interesting result to reward his reader for making it clear to the end. He doesn't always do this, but when he does, it seems like a real treat.

For example, at the end of one of his papers on number theory [E228] in which is studying numbers that are sums of two squares, he shows that 1,000,009, a number that had appeared on several lists of primes among the smallest seven-digit prime numbers, was in fact not a prime because it could be written as a sum of two squares two different ways, 1000^2+3^2 and 235^2+972^2, and he did this without giving the prime factorization of 1,000,009.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In another example of a big finish, Euler ends his first paper on the gamma function [E19] with an optimistic speculation that his "interpolation of the hypergeometric series," as he called it, could be used to define fractional derivatives, though he could not imagine what use they could possibly have.

This month's column is about a treat at the end of Euler's differential calculus book, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, "Lessons in differential calculus with its use in finite analysis and the study of series," *Calculi differentialis* for short. Euler wrote the book about 1750 and it was published in 1755. It is number 212 in Eneström's index. The title page of a 1787 edition published in Italy is shown at the right.

Euler divides his book into two parts. The first part "contains a complete explanation of this calculus" and has been translated into English by John Blanton. It has nine chapters and was 278 pages in the 1755 edition, 213 pages in the *Opera omnia*. The second part "contains the use of this calculus in finite analysis and the study of series." It is another 18 chapters, 600 pages in the original and 459 pages in the *Opera omnia*. This part has not been translated into English. The whole volume in its original is almost 900 pages long, has no illustrations, exercises or applications to "real life" problems. And it only covers differential calculus. Euler published another *three* volumes on integral calculus about 15 years later. The whole set is more than 2500 pages long. And some people complain that modern calculus textbooks are too long.

In part 2 chapters 10 and 11 of the *Calculi differentialis*, Euler tells us about maxima and minima. Chapter 15 is titled *De valoribus functionum, qui certis casibus videntur indeterminate*, "On the values of functions, which in certain cases are seen to be indeterminant." By this he means L'Hôpital's rule, though he never mentions L'Hôpital himself.

Chapter 16 has the intriguing title *De differentiatione functionum inexplicabilium*, "On the differentiation of inexplicable functions." Euler doesn't tell us what an "inexplicable function" is, only what it is not. It is not rational or algebraic or given by any of the usual transcendental functions. He gives us two examples:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x},$$

the partial sum of the harmonic series, interpolated so that x may not be a positive integer, and

$$1\cdot 2\cdot 3\cdot 4\cdots x,$$

the factorial numbers, again interpolated so that the function is defined even if x is not a positive integer. Euler began his studies of these two functions in 1729 in E19 and E20.

There is a beautiful surprise in chapter 16 while Euler is studying the function $S = 1\cdot 2\cdot 3\cdot 4\cdots x$. We would call this the gamma function today, with the small transformation that $\Gamma\left(x+1\right) = S(x)$. In one of his examples he finds an expression for $\frac{dS}{S}$ and notes that when $x = 0$ his expression gives

$$\frac{dS}{S} = -dx\cdot 0.577215664901325\,.$$

Just a page earlier he had identified this constant that begins 0.577 as the one that arises from comparing the ln n with the nth partial sum of the harmonic function, what we now call the Euler-Mascheroni constant and denote by the symbol g. Since $S(0)=1$, we can rewrite this using the modern G-notation as

$$\Gamma'(1) = -g\ .$$

This result is the highlight of Emil Artin's classic little book *The Gamma Function* [A], and gives a beautiful and unexpected link between two different objects that share the name *gamma*. Euler knew the theorem in 1755.

This would have made a fine ending for the book, but Euler had a better one. Euler used a trick that Beethoven would use fifty years later in his *Fifth Symphony*: If you have two really great endings, use them both. Use the better one second.

So we get to chapter 18, the last chapter of the second part of *Calculi differentialis*, titled *De usu calculi differentialis in resolutione fractionum*, "On the use of differential calculus in the resolution of fractions." By this, Euler means what we now call "partial fractions." Euler had introduced partial fractions in his 1748 masterpiece, *Introductio in analysin infinitorum*, "Introduction to the analysis of the infinites," [E101, E102] which he regarded as a prerequisite for calculus, so he felt confident his readers would know about them. Today we usually see partial fractions as an integration technique, though they are gradually drifting out of the curriculum. Euler will use them as an integration technique too in his integral calculus textbooks, but here he is considering them as an application of differential calculus.

He reminds us that any rational function $\frac{P}{Q}$ can be rewritten as a sum of "simple fractions" for which the denominators are either irreducible factors of the denominator Q, or powers of those factors. Euler assumes without mentioning it that the degree of P is less than the degree of Q because he thought he had made that clear in the *Introductio*. He notes that the simplest case occurs when Q is a product of distinct linear factors because then $\frac{P}{Q}$ can be rewritten as a sum where the numerators are constants and the denominators are just those same linear factors. He further reminds us of the forms that the partial fractions will have if Q has some repeated factors, some factors that are irreducible quadratics, or even repeated irreducible quadratic factors.

Rather than telling us his technique right away, Euler leads us through its derivation. He takes $\dfrac{P}{Q}$ to be his rational function and assumes that Q has a simple (not repeated) factor $f+gx$. To make sure we understand what he means, he tells us that this implies that there is a polynomial S such that $Q=\left(f+gx\right)S$ and such that $f+gx$ is not a factor of S. He writes the simple fraction that arises from the factor $f+gx$ as $\dfrac{A}{f+gx}$, (though he uses the Fraktur alphabet where we use a common "*A*") and he lumps the rest of the partial fraction expansion into the rational function $\dfrac{V}{S}$. This makes

$$\frac{P}{Q}=\frac{A}{f+gx}+\frac{V}{S}.$$

Euler solves this for V to get

$$V=\frac{P-AS}{f+gx}.$$

Now for V to be a polynomial, the denominator on the right, $f+gx$, must divide the numerator, $P-AS$, which implies that if we substitute $x=\dfrac{-f}{g}$ into the numerator, it must vanish. But when $P-AS=0$ it implies that $A=\frac{P}{S}$. This gives us a way to find A in the numerator of $\dfrac{A}{f+gx}$, which is part of the partial fraction expansion of $\dfrac{P}{Q}$.

So, when $x = \frac{-f}{g}$

$$A = \frac{P}{S} = \frac{(f+gx)P}{(f+gx)S} = \frac{(f+gx)P}{Q}.$$

To make this meaningful in modern notation, we would have to take appropriate limits, but since the meaning is clear, we will stubbornly persist in using Euler's 18th century notation. But this last expression is indeterminate of the form $\frac{0}{0}$, so what Euler taught us in chapter 15 about L'Hôpital's rule applies. We can take differentials (as they always did in the 18th century. Derivatives came later.) We get

$$A = \frac{(f+gx)\,dP + Pg\,dx}{dQ}.$$

The first term in the numerator disappears because $f + gx = 0$, leaving us with

$$A = \frac{gP\,dx}{dQ}.$$

To summarize in modern notation, when $f + gx$ is a simple factor of the denominator Q of a rational function $\frac{P}{Q}$, then we can find the coefficient A in the term $\frac{A}{f+gx}$ of the partial fraction expansion of $\frac{P}{Q}$ by taking

$$\lim_{x \to \frac{-f}{g}} \frac{gP}{dQ/dx}.$$

Note that the factor S is used in the derivation, but it does not appear in the result itself, so we don't have to divide Q by $f + gx$.

This is Euler, so of course there are examples. Example 1 is to find the coefficient corresponding to the factor $1 + x$ in the rational function $\frac{x^9}{1+x^{17}}$. Here $P = x^9$, $Q = 1 + x^{17}$, and it isn't hard to find S if we wanted to. Also $f = g = 1$. Using modern notation we get

$$\begin{aligned} A &= \lim_{x \to -1} \frac{1 \cdot \left(x^9\right)}{d\left(1+x^{17}\right)/dx} \\ &= \lim_{x \to -1} \frac{x^9}{17x^{16}} \\ &= \frac{-1}{17} \end{aligned}$$

In Euler's second example, he finds that the coefficient corresponding to the factor $1 - x$ in $\frac{x^m}{1-x^{2n}}$ is $A = \frac{1}{2n}$.

There are other examples, and there are still a few details about repeated factors and irreducible factors in the denominator, but the clever reader can probably figure out what to do. The repeated root case, for example, begins by rewriting

$$\frac{P}{Q}=\frac{V}{S}+\frac{A}{(f+gx)^{n}}+\frac{B}{(f+gx)^{n-1}}+\frac{C}{(f+gx)^{n-2}}+\text{etc.}$$

and applying L'Hôpital's rule several times.

What a nice way to end a differential calculus book, with partial fractions as an application of derivatives, rather than as an integration technique. I conducted a brief survey of 20$^{\text{th}}$ and 21$^{\text{st}}$ century English language calculus textbooks and asked a number of colleagues. Only one colleague, educated in a highly competitive university in China, had seen this in his four-semester mathematical analysis course. For the rest of us, it has been forgotten. It might be nice to remember it.

References:

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