How Euler Did It

by Ed Sandifer

Cannonball curves

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"In theory there is no difference between theory and practice. In practice there is." – Yogi Berra
also attributed to Chuck Reid, Jan L. A. van de Snepscheut, Manfred Eigen, et al.

We all know that the trajectory of a thrown object under the influence of gravity is a parabola, don't we? Isn't that one of the things that got Galileo in so much trouble back in the 1600's? Don't we do dozens of calculus problems every semester based on this fact?

This idea of a parabolic trajectory hasn't always been as obvious and widely accepted as it is today. We will devote this month's column to a brief history of trajectory curves and a bit about Euler's role in their evolution.

Early scholars of ballistics are said to have thought that a cannonball would have a triangular trajectory, though in my brief search of some 16th and 17th century books on gunnery, I could find no primary source that actually made this claim. Still, such a belief in triangular trajectories would be consistent with the scientific views of the time. Remember that we have not always classified chemical elements with the Periodic Table. Once there were only four elements, Earth, Air, Fire and Water. We could explain natural phenomena by the tendency of an object to seek to restore its proper balance of these four elements.

Some phenomena are difficult to explain within the four-element system. For example, if Water is an element, its natural balance ought to be just Water. But then, how can water contain enough of the element Air to evaporate? Fortunately, there were enough taverns that this question could be thoroughly discussed and a satisfactory solution found.

Some phenomena had explanations that seem quite silly today. For example, women were said to be overly prone to crying. This was because they had a tendency to accumulate too much Water, and tears were the natural way for women to re-balance their elements by removing the excess Water.

To the modern reader, such explanations seem strange and unfamiliar, sometimes even incredible, but in their time they were remarkably useful in explaining and even predicting natural phenomena.

In the case of ballistics, here is an argument in favor of a triangular trajectory. Suppose we propel an object like a cannonball into the air. A cannonball is mostly Earth when its elements are correctly balanced. When we propel it into the air, though, we are adding Air to its composition, so it rises. If we could make the Air stay in the cannonball, that would be unnatural, but we could do it with witchcraft. That's one of the things witchcraft could do; force things into an unnatural state, like floating cannonballs. However, our cannonball obeys the laws of nature, and it expels its excess Air. When the Air is all gone, balance is restored, and the cannonball falls straight down, as is its nature.

By the end of the 1500's, Nicolo Tartaglia (famous for his feud with Cardano over the algebraic solution of cubic equations) had come to doubt this triangular trajectory theory. He wrote a book on the theory of ballistics. At the behest of King Henry VIII, that book was promptly translated into English [T] and published in 1588 with a title that began *Three bookes of colloquies concerning the arte of shooting of great and small peeces of artillerie*. The illustration above shows how Tartaglia thought that a cannonball's trajectory would begin almost straight and would gradually turn downward, rather than making the sudden turn downward described by the triangular theory. Tartaglia's argument is wordy and legalistic. In the style of his times, he thinks that truth will be found by a careful examination of the reasons more than by experimentation or observation. Tartaglia presents reasons in favor of, and opposed to the idea of curved trajectories, almost in the style of a courtroom prosecution and defense, and the reader is expected to act as the judge

One particularly interesting aspect of Tartaglia's argument is that he describes the curve as being "straighter" between *A* and *F* than it is between *A* and *C*, and that, if points were taken close enough together, the curve could be regarded as straight between those two points. He doesn't actually say it, but the modern reader sees him as coming close to describing a curve as a collection of straight line elements an idea that wasn't actually articulated for at least a hundred years.

The illustration at the right, also from Tartaglia's *Colloquies*, has the colorful motto *Scientia non habit inimicum prater Ignoratem*, "Science has no enemies except Ignorance." It shows a more elevated trajectory than the one above, and the text tells us that the fireball will fall straight down upon its target.[1] Though the trajectory is clearly a curved one, it cannot be a parabola.

Anno domini. 1588

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[1] One wonders if the victims whose homes are about to be burned think that the science that calculated this trajectory might be their enemy?

In the early 1600's, Galileo gave us the parabolic trajectory. His story is well known, so we won't repeat it here. I particularly like Berthold Brecht's version.[B] His play, *Galileo*, gives a moving account of the political, scientific and religious issues surrounding Galileo's discoveries, though some think that Brecht is a bit too quick to "bend the truth" to make a good story.

Most of our trajectory problems include a disclaimer like "ignoring air resistance." What happens, though, if air resistance *is* significant, and ought not be neglected? Enter Euler.

Euler wrote a book and three articles on ballistics. Though his contributions are relatively few, they were extremely influential, and they have an interesting story. We begin that story in 1736, when Euler published his two-volume masterwork of physics, the *Mechanica*. Euler worked out in complete detail the mechanics of point masses that Newton had only hinted at. English mathematicians and scientists interpreted Euler's work as criticism of their hero Newton, and some of them responded with bitterness and hostility.

One Englishman, Benjamin Robins, was particularly vitriolic. [R1] A few years later Euler's new employer, Frederick the Great, asked Euler what the best book on mathematical ballistics was. Despite the bad review Robins had given his own book, Euler recommended Robins' book and agreed to translate it from English into German. Euler's "translation" came out in 1745 and is remarkable on a number of levels. First, there is no evidence that Euler knew any English. Second, Robins' book had been 150 pages long. Once Euler got done with adding his comments, it was 720 pages long. In 1777, Hugh Brown translated the book back into English, and in 1783 it was translated into French by someone named Lombard. Napoleon supposedly read that edition and it is one of the things that influenced him to rely so much on his scientists, engineers and mathematicians.

Robins' book of 1742 [R2] contains the passage illustrated below:

(87)

PROP. VI.

The Track described by the Flight of Shot or Shells is neither a Parabola, nor nearly a Parabola, unless they are projected with small Velocities.

How did Robins know that the trajectory of a cannon ball was not a parabola? He had invented a device called a *ballistic pendulum* that measured the speed at which a cannon ball left the barrel of the cannon. It was quite a clever and simple device, and my father tells me that when he was younger, every high school physics student in Oklahoma who owned a gun did a laboratory experiment to find the muzzle velocity of his own rifle. Can you imagine doing such an experiment in today's schools?

Robins also wrote:

So much for the parabola. Robins knew that Galileo's claim did not apply to objects moving as fast as cannon balls travel, and he put it in his book. Robins didn't say much about what the trajectory really was, and on this, Euler did not add much in his translation.

Euler revisited the question of trajectories in 1753 in a 40-page article with a title that translates as "Research on the true curve that is described by bodies shot through the air or in any other fluid." [E217] Euler reports that there is theoretical and experimental evidence that air resistance should be proportional to the square of the body's speed through the air. He tells us that Newton had written down the differential equations for such trajectories, but that he had "uselessly tried various ingenious methods to arrive at a solution." He says that his own teacher Johan Bernoulli had been the first to give a solution to the problem.

Euler identifies three forces always acting on a projectile:

1. the accelerating force of gravity, always directed vertically downward
2. the buoyant force of the fluid, always directed upward
3. the resistance of the fluid, always directed against the direction of the motion

It is this last force that Euler assumes is proportional the square of the velocity. The second force is usually small, but Euler has two reasons to consider it. First, he wants his results to be general enough to describe trajectories through *any* fluid, including water, where buoyancy is significant. Second, he knows from some of his other work that the density of air changes with altitude and weather conditions, and he wants to take that into account.

Euler sets out to find what he can about the nature of the trajectory. He works from Figure 1 below.
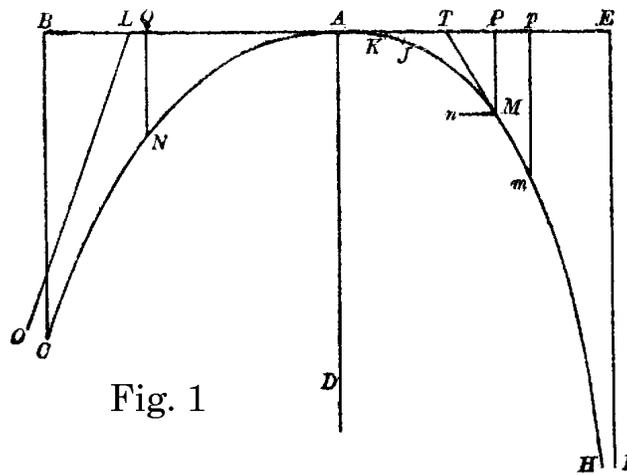
Fig. 1

Euler divides his discussion into two "branches" of the curve, the *ascending branch*, *CNA*, and the *descending branch*, *AMH*. He finds that the *x* component of the velocity is monotonically decreasing (though he doesn't use those words) and that the descending branch has a vertical asymptote, shown in Figure 1 as *EF*. This alone would show that the curve is not a parabola, but Euler adds two other facts that show the descending branch is not a parabola. First, the point at which the curve has its greatest curvature is not at the vertex *A*, but is a point on the descending branch near *A*, labeled *K* in Figure 1. He also shows that the point where the velocity is minimum is not at *A*, either, but at a point beyond *K*, here labeled *J*.

Next he moves to the ascending branch and shows that when the curve is extended past its initial point at *C*, it, too, has an asymptote. This asymptote, though, is diagonal, shown in Figure 1 as line *LQ*. This gives further evidence that the trajectory is not a parabola, for a parabola cannot have a diagonal asymptote either. Also this curve cannot be symmetric, with one vertical asymptote and one diagonal one.

Euler finishes this paper with some tables that show how to find the true curve accurately and how to find various things of interest to artillerymen, like the height of the vertex and the speed of the projectile at various points along its trajectory.

Euler's results are essentially correct, though he does not know about what we now call the Magnus force, and some other forces that turn out to be significant. Let's compare Euler's trajectory to the trajectories that people had proposed earlier.

It's not a parabola. Euler takes pains to show that. Galileo didn't get it.

It's not a triangle. The shape allegedly proposed by the alchemists isn't right either.

It travels diagonally for a while, then curves, and falls almost straight down. Tartaglia, with his ideas of Earth, Air, Fire and Water, came the closest. Of course, it helped that he looked closely at what really happens before he made his predictions.

Let's hear it for four elements.

References:

[B]   Brecht, Berthold, *Galileo*, Charles Laughton, tr., NY, Grove Press, 1966.

[E77]   Euler, Leonhard, *Neue Grundsätze der Artillerie aus dem Englischen des Herrn Benjamin Robins übersetzt und mit fielen Anmerkungen versehen*, Berlin, 1745. Reprinted in *Opera Omnia* Series II vol 14 p. 3-409. This is one of the few works of Euler that is not yet available through The Euler Archive at www.EulerArchive.org. If you have a copy of the original, I am sure they would like to hear from you.

[E217]   Euler, Leonhard, Recherches sur la veritable courbe qui décrivent les corps jettés dans l'air ou dans un autre fluide quelconque, *Mémoires de l'académie des sciences de Berlin*, **9** (1753) 1755, p. 321-362. Reprinted in *Opera Omnia* Series II vol 14 p. 413-447. Available through The Euler Archive at www.EulerArchive.org.

[R1]   Robins, Benjamin, *Remarks on Mr. Euler's Treatise of Motion, Dr. Smith's compleat System of Optics, and Dr. Jurin's Essay on Distinct and Indistinct Vision*, London, 1739

[R2]   Robins, Benjamin, *New principles of gunnery: containing the determination of the force of gun-powder, and an investigation of the difference in the resisting power of the air to swift and slow motions.* London, J. Nourse, 1742. Available online through *Early English Books Online*.

[T]   Tartaglia, Nicolas, *Three bookes of colloquies concerning the arte of shooting of great and small peeces of artillerie*, John Harrison, London, 1588. Available online through *Early English Books Online*.

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