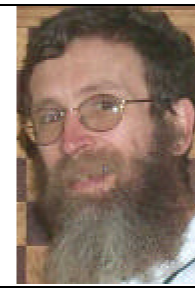
How Euler Did It

by Ed Sandifer



Foundations of Calculus

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As we begin a new academic year, many of us are introducing another generation of students to the magic of calculus. As always, those of us who teach calculus are asking ourselves again, "What is the best way to begin calculus?" More specifically, "How do we start to teach students what a derivative is?" Some of us will start with slopes and others choose limits. Among those who begin with limits, some will use epsilons and deltas and others will use a more intuitive approach to the algebra of limits. A few might use non-standard analysis, as rigorously presented in the wonderful book [K] by Jerome Kiesler. Newton began with "fluxions," while Leibniz used differentials and a "differential triangle." Regular readers of this column, though, know to ask "How did Euler do it?"

Almost as soon as it was invented (or, if you prefer, discovered) people began arguing about its foundations. Leibniz looked for an algebraic basis for calculus, while Newton argued in favor of geometric foundations. The controversy continued for more than a hundred years, with key contributions from Berkeley and Lagrange, until most of the issues were finally resolved in the time of Cauchy, Riemann and Weierstrass.

Euler published his differential calculus book, *Institutiones calculi differentialis,* [E212], in 1755. The book has two parts. Euler describes the first part, nine chapters, 278 pages in the original, as "containing a complete explanation of this calculus." John Blanton translated this part of the book into English in 2000, and most of the quotations used in this column are from John Blanton's edition. The second part of the book, 18 chapters, 602 pages, "contains the use of this calculus in finite analysis and in the doctrine of series." A translation of this part of the book has not yet been published, though there are rumors that people are working on it.

When Euler sat down to write the *Calculus differentialis,* as it is commonly called, he had to decide how to explain the foundations of calculus and the reasons calculus "works." In his preface he writes (in John Blanton's translation, p. vii.):

> [D]ifferential calculus … is a method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment."

This echoes with Newtonian sentiments. "Vanishing increments" sound like Newton's "evanescent quantities," and are open to Berkeley's sarcastic barbs, calling derivatives "ghosts of departed quantities." Euler, who learned his calculus from Johann Bernoulli, a follower of Leibniz, understands these criticisms, and in the very next paragraph he writes

> "[D]ifferential calculus is concerned not so much with vanishing increments, which indeed are nothing, but with the ratio and mutual proportion. Since these ratios are expressed as finite quantities, we must think of calculus as being concerned with finite quantities."

Later (p. viii.) he writes

> "To many who have discussed the rules of differential calculus, it has seemed that there is a distinction between absolutely nothing and a special order of quantities infinitely small, which do not quite vanish completely but retain a certain quantity that is indeed less than any assignable quantity."

Euler seems to want it both ways. He wants to use infinite numbers, usually denoted i or n, as well as infinitesimals (he calls them "infinitely small,") usually denoted w. He wants to take their ratios, add, subtract and multiply them as if they matter, and then throw them away when it suits his purposes. It is exactly the behavior that Berkeley was trying to discourage and that Cauchy and Weierstrass eventually repaired.

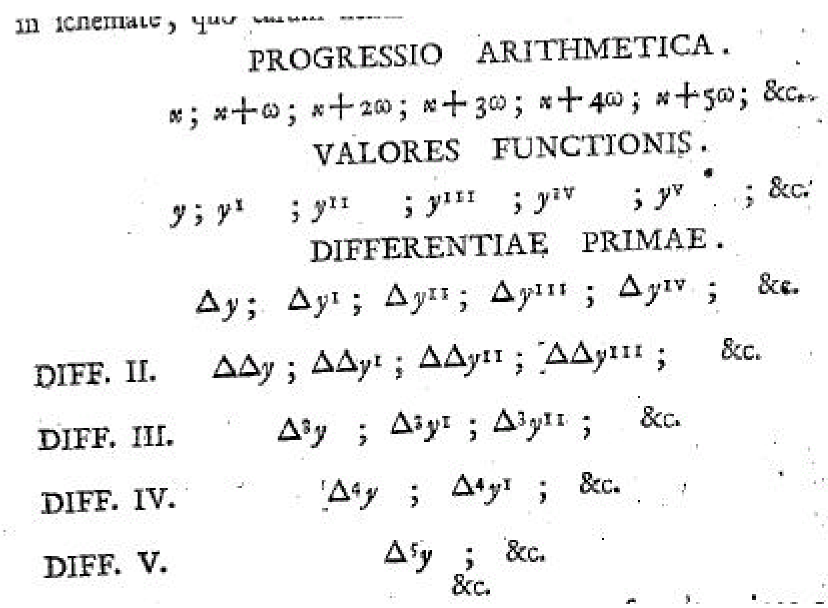
Now that we've seen these philosophical underpinnings, let's look at how Euler teaches us calculus.

Euler's Chapter 1 is "On finite differences" (*De differentiis finites*.) Euler gives us a variable quantity x, and an increment w. For now, w is assumed to be finite. He asks how substituting $x + w$ for x in a function "transforms" that function, and gives us the example

$$\frac{a+x}{a^2+x^2} \text{ is transformed into } \frac{a+x+w}{a^2+x^2+2xw+w^2}.$$

The value x and its increment w give an arithmetic sequence, x, $x + w$, $x+ 2w$, $x + 3w$, etc., and these, in turn, transform a function y into a sequence of values that he denotes y, y^{I}, y^{II}, y^{III}, etc.

With this notation in place, he is ready to describe the first, and higher differences of the function. This is apparently the first time the symbol Δ for this purpose. The image below, from The Euler Archive, shows Euler's definition of the higher differences:



in schemate, quo ...

PROGRESSIO ARITHMETICA.

$x;\ x+\omega;\ x+2\omega;\ x+3\omega;\ x+4\omega;\ x+5\omega;$ &c.

VALORES FUNCTIONIS.

$y;\ y^{I};\ y^{II};\ y^{III};\ y^{IV};\ y^{V};$ &c.

DIFFERENTIAE PRIMAE.

$\Delta y;\ \Delta y^{I};\ \Delta y^{II};\ \Delta y^{III};\ \Delta y^{IV};$ &c.

DIFF. II. $\Delta\Delta y;\ \Delta\Delta y^{I};\ \Delta\Delta y^{II};\ \Delta\Delta y^{III};$ &c.

DIFF. III. $\Delta^{3} y;\ \Delta^{3} y^{I};\ \Delta^{3} y^{II};$ &c.

DIFF. IV. $\Delta^{4} y;\ \Delta^{4} y^{I};$ &c.

DIFF. V. $\Delta^{5} y;$ &c.

&c.

Next we get some of the elementary properties of higher differences, starting with their values in terms of the values of *y*.

$$\Delta\Delta y = y^{II} - 2y^{I} + y$$
$$\Delta\Delta y^{I} = y^{III} - 2y^{II} + y^{I}$$
$$\Delta^{3} y = y^{III} - 3y^{II} + 3y^{I} - y.$$

and so on up to 5th differences. He follows these with rules for sums and products:

If $y = p+q$ then $\Delta y = \Delta p + \Delta q$, and

If $y = pq$ then $\Delta y = p\Delta q + q\Delta p$.

By the end of this 24-page chapter, Euler has taught us to find differences of sums, products and radicals involving polynomials, sines, cosines, logarithms and radicals, as well as inverse differences, finding a function that has a given first difference.

In Chapter 2 Euler uses differences to study what he calls "series." We would call them "sequences." Probably his most interesting result in the chapter is the discrete Taylor series. Consider a sequence a, a^{I}, a^{II}, etc., with indices 1, 2, 3, 4, etc. Let the first, second, third, etc. differences be *b, c, d,* etc. Then the "general term" of index *x* is given by

$$a+\frac{(x-1)}{1}b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\text{etc.}$$

Euler's chapter 3, "On the Infinite and the Infinitely Small," he returns to the philosophical underpinnings of calculus. The chapter is 30 pages long, and is mostly a fascinating philosophy-laced essay on the nature of infinites and infinitesimals in the real world. In the 18^{th} century, real numbers were not the free-standing axiomatized objects they became late in the 19^{th} century (thanks to Dedekind and his cuts). The properties of the real numbers were expected to reflect the properties of the real world they describe. Hence, the debate between Newton and Leibniz about whether real world objects are infinitely and continuously divisible (Newton), or composed of indivisible ultimate particles (that Leibniz called *monads*) was also a dispute over the nature of the real numbers. That, in turn, became a dispute about the nature and foundations of calculus. Though their public argument was mostly about who first discovered calculus, each also believed that the other's version of calculus was based on false foundations.

After careful consideration, Euler eventually sides mostly with Newton and accepts the "reality" of the infinite and infinitely small. The difficult part of this is that, if dx is an infinitely small quantity, then

> "Since the symbol ∞ stands for an infinitely large quantity, we have the equation
>
> $$\frac{a}{dx}=\infty.$$
>
> The truth of this is clear also when we invert:
>
> $$\frac{a}{\infty}=dx=0\text{ ."}$$

So, Euler is stuck with the paradox that the quantity dx is, in some sense, both zero and not zero. He cannot resolve this paradox, so he has to figure out a way to avoid it.

To do this, he begins to use his results on differences from the first two chapters. When he takes a sequence with an infinitely small quantity dx as its difference, then considers second differences, he is forced to conclude that "a/dx^2 is a quantity infinitely greater than a/dx," and similarly for higher differences. "We have, therefore, an infinity of grades of infinity, of which each is infinitely greater than its predecessor."

Essentially, he introduces some rules for the use of infinite and infinitesimal quantities, roughly equivalent to our techniques for manipulating limits. A quantity like $a/dx : A/dx$, where a and A are finite quantities (i.e., neither infinite nor infinitesimal) should not be resolved by first dividing by dx, for that leads to ∞/∞. This cannot be evaluated because, as he noted above, there are many different sizes of infinity, and this expression doesn't tell us which size we have. Instead, the quantity $a/dx : A/dx$ should first be reduced to *a/A*, a ratio of finite quantities.

Euler tells us that also “it is possible not only for the product of an infinitely large quantity and an infinitely small quantity to produce a finite quantity, … but also that a product of this kind can also be either infinitely large or infinitely small.”

The last part of chapter 3 deals with issues of convergence. It could be the basis of some future column.

Now that Euler believes he has convinced us of the logical integrity of his foundations, he returns to his calculations with series and differences. He reminds us that, given a variable x, a quantity y that depends on x, and an increment $\Delta x = w$, then y has a difference of the form

$$\Delta y = y^{\mathrm{I}} - y = Pw + Qw^2 + Rw^3 + Sw^4 + \text{etc.}$$

Taking $w = dx$, we get that $dy=Pdx$, or, as we would write it today, $\frac{dy}{dx} = P$. Note that Euler and his contemporaries did calculus with differentials, dy and dx, and not with derivatives, $\frac{dy}{dx}$. Euler explains how the higher coefficients, Q, R, S, etc., are related to higher differences, and he’s ready to go with the rules of calculus.

Before he goes, though, he makes a remark to appease his Leibnizian friends by criticizing Newton’s fluxion notation. Newton would write $\dot{y}$, $\ddot{y}$ or $\dddot{y}$ where Euler would write P, Q or R, or maybe dy, d^2y or d^3y. Euler writes that Newton’s notation “cannot be criticized if the number of dots is small … On the other hand, if many dots are required, much confusion and even more inconvenience may be the result.” As an example, he gives (and here, for the first time, we don’t follow Blanton’s translation) “The tenth fluxion, though would be very inconveniently represented by $\overset{\vdots\vdots}{y}$, and our notation of $d^{10}y$ is much easier to understand.”

With this, Euler sets out to give the usual rules of differential calculus, of course using differentials instead of derivatives and, in part one, omitting all applications. This column is about Euler’s foundations of calculus, so we will leave out most of the content, for now. One of Euler’s examples, though, is particularly elegant, from chapter 6, “On the Differentiation of Transcendental Functions:”

“If $y = e^{e^{e^x}}$, then

$$dy = e^{e^{e^x}} e^{e^x} e^x dx.”$$

It is visually striking if you write it on a blackboard and use a longer string of e’s.

Euler is sometimes criticized by modern mathematicians for what seems like a reckless use of infinite and infinitesimal numbers in his calculations, and for ignoring the foundations of calculus. What he writes in the *Calculus differentialis*, though, makes it clear that he was very aware of the issues involved, and that he tried hard to resolve them. In fact, he himself believed that he had indeed put calculus on a solid philosophical foundation.

Two generations after Euler, though, the way we build the foundations of mathematics changed, and a philosophical basis was no longer accepted. The age of Cauchy and Weierstrass sought less geometric and more axiomatic foundations, and Euler's approach was discarded as insufficiently rigorous.

In the 20th century, though, Abraham Robinson developed non-standard analysis, and showed how Euler's techniques could be made rigorous. Jerome Keisler, [K] in turn, used Robinson's constructions to write a modern calculus text. It took 220 years, but Euler's *Calculus differentialis* was eventually shown to have rigorous foundations. Now, if we want to, we can do the way Euler did it.

References:

[E212] Euler, Leonhard, *Insitutiones calculi differentialis cum ejus usu in analysi finitorum ac doctrina serierum*, St. Petersburg, 1755. Reprinted in *Opera Omnia*, Series I vol 10. English translation of chapters 1 to 9 by John Blanton, Springer, New York, 2000. Original is available online through The Euler Archive at EulerArchive.org.

[K] Keisler, H. Jerome, *Elementary Calculus*, Prindle, Weber & Schmidt, Boston, 1976. The entire book is available (Free! under a Creative Commons License) at www.math.wisc.edu/~keisler/calc.html.

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How Euler Did It is updated each month.