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How Euler Did It

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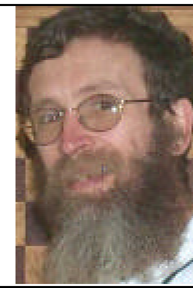

Formal Sums and Products

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Two weeks ago at our MAA Section meeting, George Andrews gave a nice talk about the delicate and beautiful relations among infinite sums, infinite products and partitions. Dr. Andrews, the Evan Pugh Professor of Mathematics at Penn State, and is known even among non-mathematicians for his 1976 discovery of "Ramanujan's Lost Notebook," a collection of 138 pages of notes that had lain unnoticed in the archives at Trinity College, Cambridge. A nice account of the colorful history of the "Lost Notebook" is online at [B].

Dr. Andrews described some of the tools he uses to understand and to extend Ramanujan's work. They include functions called *q-series,* [W] defined and denoted by

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right).$$

One q-series is particularly important, and is known among friends of Ramanujan and Andrews as Euler's function:

$$f(q) = (q;q)_\infty = \prod_{k=1}^{\infty}\left(1-q^k\right).$$

As we will see (and as long-time readers of this column have already seen in [S 2005],) when these products are expanded into a sum, then the coefficients of the resulting series contain information about partitions.

This month's column will discuss some of the tools Euler developed to understand the relations among infinite products, infinite sums and combinatorics that make the work of Andrews and Ramanujan so beautiful.

Euler danced among products, sums and combinations several times. In [E19] he used products to discover the Gamma function. In [E41], he linked products and sums to solve the Basel problem. Then in [E158] he linked sums, products and partitions to solve Philip Naudé's problem. He touched on the connections several other times, but rather than trace how he developed the ideas, we will look at his unified presentation in Chapter 15 of the *Introductio in analysin infinitorum* [E101], the chapter titled "On series that arise from the expansion of products."

Let us recall when we first learned about quadratic equations. We learned that if a product, $(x-r)(x-s)$ expands into a sum, x^2+bx+c, then the roots of the quadratic are *r* an *s*, that $c = rs$ and that $b = -(r + s)$.

Euler learned from a slightly different book. He learned this same fact using a product of the form $(1+a\,z)(1+b\,z)$ and a quadratic $1+Az+Bz^2$. Though this makes the roots a little harder to write, it makes the sums and products a little easier, so that $A=a+b$ and $B=ab$.

Euler opens Chapter 15 by generalizing this result to many factors. He asks us to consider the sum and product

$$1+Az+Bz^2+Cz^3+\text{etc.}=(1-a\,z)(1-b\,z)(1-g\,z)\,\text{etc.} \qquad (1)$$

Then he explains that

$A=a+b+g+d+e+z+$etc. = the sum of the coefficients taken individually
$B=ab+ag+bg+ad+bd+gd+$ etc. = the sum of the products taken two at a time
C = the sum of the products taken three at a time,
D = the sum of the products taken four at a time,
etc.

Euler is deliberately vague about whether he means these sums and products to be finite or infinite. This is partly because he, like most others of his time, believed in something called the "Principle of Continuity." This was a philosophical principle of Leibniz, and stated, roughly, that if two things are not much different, then their effects and properties will not be much different, either. [P] Leibniz captured the principle with his aphorism "Nature makes no leaps." Thus, since the relations between factors and coefficients that are true for finite sums and products should also be true for infinite ones.

Euler also ignored all issues of convergence in this discussion, unlike today when we wave our hands and either mumble "formal power series" or "suitable radius of convergence."

For his first example, Euler takes the Greek letters a, b, g, d, etc., to be the sequence of prime numbers, 2, 3, 5, 7, 11, 13, etc. Now we'll do in three or four steps what Euler does in one step.

Consider the product

$$(1+2z)(1+3z)(1+5z)(1+7z)(1+11z)\,\text{etc.}$$

Equation (1) and the properties of *A*, *B*, *C*, etc. tell us that this product expands to give

$$\begin{aligned}1&+(2+3+5+7+11+13+\text{etc.})z\\&+(2\cdot3+2\cdot5+3\cdot5+2\cdot7+3\cdot7+5\cdot7+\text{etc.})z^2\\&+(2\cdot3\cdot5+2\cdot3\cdot7+3\cdot5\cdot7+2\cdot3\cdot11+\text{etc.})z^3\\&+(\text{etc.})z^4+\text{etc.}\end{aligned}$$

Euler takes $z = 1$, performs the multiplications, rearranges the terms into increasing order, and names the whole sum P:

$$P = 1 + 2 + 3 + 5 + 6 + 7 + 10 + 11 + 13 + 14 + 15 + 17 + \text{etc.}$$

"in which series," he tells us, "all natural numbers occur except those that are powers, and those which are divisible by some power." We would call this the series of "square-free" numbers, those that are not divisible by any square number except 1.

Note that Euler isn't really interested in the *value* of this series; rather he wants us to see the numbers that occur in the series itself.

For his next example, he takes his coefficients a, b, g, d, etc. to be "any power of the prime numbers." His formula makes it clear that he means the power to be negative, as he writes

$$P=\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{etc.}$$

In many ways, this is a lot like his previous example. This formula expands into a series to give

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\text{etc.}$$

where again the denominators are based on square-free numbers. It is a more interesting example than it seems to be. First, it is true. For $n > 1$, both the series and the product converge, and they converge to the same (finite) value.

Second, it is an example of the Principle of Continuity applied in reverse. We had a fact about infinite series that do not converge. The argument that convinced the reader (maybe) of that fact also applies to the closely related infinite series that *do* converge. If the first argument convinced us, then this argument should convince us, as well.

Third, Euler has a plan. These examples really are going somewhere. He's not telling us where, though.

His next example has a new idea, but no clues where he's going. Take the negatives of the powers in the previous example, so that

$$P = \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{etc.} \qquad (2)$$

The formula tells us that this expands to give

$$P = 1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}+\text{etc.} \qquad (3)$$

where the denominators again include only the square-free bases, but the signs are determined by the number of prime divisors in the denominator. Those with an odd number of prime divisors, like the primes themselves, or like $30 = 2\cdot 3\cdot 5$, have the negative sign, but those with an even number of divisors, like $6 = 2\cdot 3$, $10 = 2\cdot 5$ and $15 = 3\cdot 5$, have a positive sign.

Remember these formulas. We will see them again later.

Now we turn to quotients. Euler asks us to "now consider this expression"

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-z\delta)(1-\varepsilon z)\text{etc.}}.$$

Again he expands the quotient into a series

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.}$$

and, without explanation, tells us that

A = the sum of the coefficients taken one at a time
B = the sum of the coefficients multiplied together two at a time, possibly with repetition
C = the sum of the coefficients multiplied together two at a time, again with repetition allowed
D = the sum of products taken four at a time,
etc.

Let's fill in a few steps that Euler omitted, and give a simple example. Suppose there are only two factors in the denominator, so our quotient is

$$\frac{1}{(1-\alpha z)(1-\beta z)} = \frac{1}{(1-\alpha z)}\cdot\frac{1}{(1-\beta z)}$$

Each of the factors on the right expands into a geometric series;

$$\frac{1}{1-\alpha z} = 1+\alpha z+\alpha^2 z^2+\alpha^3 z^3+\alpha^4 z^4+\text{etc.},$$

and likewise for the other factor,

$$\frac{1}{1-bz} = 1 + bz + b^2z^2 + b^3z^3 + b^4z^4 + \text{etc.}$$

We note in passing that Euler gets these geometric series identities by "actual division" of the quotient, and not from manipulations on the series. In other words, he finds the series given the quotient, instead of doing what we usually do today start with the series and find its analytical value.

When we multiply together these two series, we get

$$\frac{1}{(1-az)(1-bz)} = 1 + (a+b)z + (aa+ab+bb)z^2 + \left(a^3 + a^2b + ab^2 + b^3\right)z^3 + \text{etc.}$$

We see that the coefficients *A, B, C,* etc., are as Euler described.

Euler's next example is deceptively simple. He takes a quotient with just one factor, $a = 1/2$ and sets $z = 1$, to get

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.}$$

Note that Euler is thinking of this as expanding the quotient, not as summing the series, and that he's interested in the terms of the series, not the value of the sum.

Euler proceeds to the case $a = ½$, $b = ⅓$ and claims

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \text{etc.},$$

where the denominators only involve numbers with no prime divisors other than 2 and 3.

This begs to be extended, so Euler takes $a, b, g, d,$ etc., to be the sequence of prime numbers, 2, 3, 5, 7, 11, 13, etc., and $z = 1$. Then he writes (by the Principle of Continuity) that if

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{etc.}}$$

then

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.}$$

This last is the harmonic series, a series we know diverges. Euler would say that its value is $\ln \infty$. If we write this in modern notation, using Sigma for sums and Pi for products, we get

$$\sum_{k=1}^{\infty}\frac{1}{k}=\prod_{p\text{ prime}}\frac{1}{1-\frac{1}{p}},$$

the famous Sum-Product formula for the Riemann Zeta function. This formula has a prominent place on the cover of William Dunham's book [D] *Euler: The Master of Us All*. Still, the formula isn't really *true*. Since the harmonic series diverges, it can't be said to have a real value, so its value can't really be equal to anything else.

Again, the Principle of Continuity has something to add. Euler goes on to tell us that if

$$P=\frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right)\text{etc.}},$$

then, by the same calculation,

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{9^n}+\text{etc.}$$

Again, in modern notation, this is

$$\sum_{k=1}^{\infty}\frac{1}{k^n}=\prod_{p\text{ prime}}\frac{1}{1-\frac{1}{p^n}}$$

and this time, for $n>1$, both the sum and the product converge, and they are equal. The sum on the left is Riemann's Zeta function, and this fact is one of the fundamental properties of the Zeta function.

This is the second time Euler proved this formula. We saw the first proof a few months ago [S 2006] when we were looking at some results from [E72]. This proof is quite different.

This chapter of the *Introductio* goes on quite a bit farther, using these techniques to calculate values of particular series, but we will wrap it up with one last result, a lesser-known property of the Zeta function.

As before, take

$$P=\frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right)\text{etc.}}$$

so that P is the Zeta function

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{9^n}+\text{etc.}$$

Now, take Q to be the reciprocal of P, so that

$$Q = \left(1-\tfrac{1}{2^n}\right)\left(1-\tfrac{1}{3^n}\right)\left(1-\tfrac{1}{5^n}\right)\left(1-\tfrac{1}{7^n}\right)\left(1-\tfrac{1}{11^n}\right)\left(1-\tfrac{1}{13^n}\right) \text{etc.}$$

This is the same product we saw in formula (2), (did you remember it like we told you to?) except it's named Q now instead of P. So, we know from formulas (2) and (3) that this product expands as a series to give

$$Q = 1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}+\text{etc.}$$

From the product forms, it is completely obvious that $PQ = 1$, but as series, the fact is quite remarkable.

There is always something interesting to learn from the *Introductio*. Ramanujan appreciated this kind of analysis, and people like George Andrews remind us that we still have much to learn by following the footsteps of Euler and Ramanujan.

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How Euler Did It is updated each month.