How Euler Did It

by Ed Sandifer

19th Century Triangle Geometry

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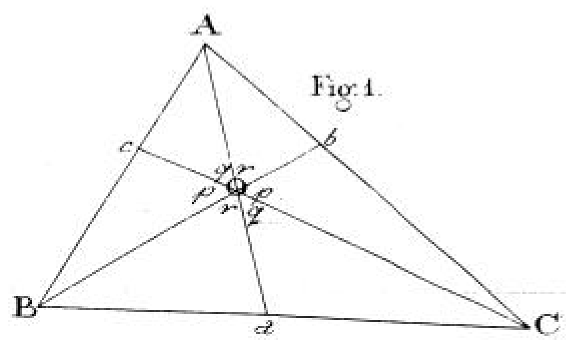
When we read about Euler, or about any other historical figure, we must remember that he lived in his own times. The 18th century was very different from the 21st in ways that we hardly ever think about. There are the obvious differences; now we have Internet, iPods, airplanes and automobiles. I am fond of reminding my students that we also have indoor plumbing, grocery stores and paper money. So, when we read Euler, we must try to understand how the problems he works on and the techniques he uses are embedded in his own times, and not in ours. He was speaking to and writing for an 18th century audience and we are lucky that the things he was saying are still useful and interesting today. So, when we find Euler seeming to use 17th century techniques to solve a 19th century problem, we might raise an eyebrow.

Euler's paper *Geometrica et sphaerica quaedam,* [E749] which translates uninformatively as "Certain geometric and spheric things," is such a paper. Its main result is a theorem in triangle geometry, a subject that was extremely popular and important in the late 19th century. The main results in triangle geometry are summarized in excellent books like that of Coxeter and Greitzer [C+G].

In contrast, one of the major mathematical themes of Euler's era was the gradual evolution, led by Euler himself, from a mathematics based on techniques and objects of geometry to one based on algebra and analysis. Though this is not the place to dwell too much on this point, we note that at the beginning of the 18th century, mathematicians called themselves "geometers" and they used calculus to study curves, in the style of L'Hôpital. By the end of the century, they called themselves "mathematicians" and "analysts" and used calculus to study functions. In E749, Euler gives three proofs of a 19^{th} century result, but his third proof, clearly his favorite of the three, is a proof with a 17^{th} century flavor.

Euler apparently wrote E749 in 1780. Euler's son-in-law Nicolas Fuss, presented it to the Academy in St. Petersburg, along with three other papers, on May 1 of that year. In 1780, Euler was 73 years old and he no longer attended the meetings of the Academy himself. Euler's last meeting seems to have been on January 16, 1777, after which Euler sent his papers in to the Academy with his assistants. In 1780, Euler had been blind for almost 15 years, and he had a team of assistants to whom he dictated hundreds of manuscripts. One of the portraits of Euler, shown above, has a sub-portrait, a smaller rectangle beneath the oval of the main portrait. The sub-portrait shows two men, one with pen and paper, sitting at a table. Apparently it pictures Euler dictating to one of his assistants, probably his son, Johan Albrecht, because Euler himself could no longer read or write.

Let us turn to the mathematics. Euler gives us the triangle ΔABC shown in Fig. 1, cut by concurrent segments Aa, Bb and Cc, where points given by lower case letters are on the sides opposite the vertices given by the upper case letters. Note that the point where the segments intersect is named O. Euler asks, given the lengths of the segments AO, Oa, BO, Ob, CO and Oc, can he reconstruct the triangle?



He finds that there will not be such a triangle unless certain conditions on the ratios of the lengths of the segments are satisfied, and gives us the following:

Theorem: If in any triangle ABC are drawn from each angle to the opposite side any straight lines Aa, Bb, Cc cutting each other at a common point O, then they will always satisfy this property, that

$$(1) \qquad \frac{AO}{Oa}\cdot\frac{BO}{Ob}\cdot\frac{CO}{Oc}=\frac{AO}{Oa}+\frac{BO}{Ob}+\frac{CO}{Oc}+2$$

Euler's proof is rather long and not very elegant. We debated omitting it but eventually decided to include it so that later we could admire how much more elegant his third proof turns out.

Proof: Take $AO=A$, $BO=B$, $CO=C$, $Oa=a$, $Ob=b$, $Oc=c$ and label the six angles around O as shown. Note that $p + q + r = 180°$. As was typical in his times, Euler expects us to be able to use context to distinguish the points A, B, C, a, b and c from the lengths with the same names.

We can find the area of ΔAOc to be $\Delta AOc = \frac{1}{2} Ac \sin q$. Similarly, $\Delta BOc = \frac{1}{2} Bc \sin p$ and $\Delta AOB = \frac{1}{2} AB \sin(p+q)$.

Since $\sin(p+q) = \sin r$, and since the areas of the first two triangles sum to the third, we get:

$$AB \sin r = Ac \sin q + Bc \sin p .$$

Similarly, for the other two pairs of triangles, we get

$$BC \sin p = Ba \sin r + Ca \sin q$$
$$CA \sin q = Cb \sin p + Ab \sin r$$

Dividing these equations by *ABc, aBC* and *AbC* respectively gives

$$(2) \qquad \begin{aligned} \frac{\sin r}{c} &= \frac{\sin q}{B} + \frac{\sin p}{A} \\ \frac{\sin p}{a} &= \frac{\sin r}{C} + \frac{\sin q}{B} \\ \frac{\sin q}{b} &= \frac{\sin p}{A} + \frac{\sin r}{C} \end{aligned}$$

Euler pauses to point out the pattern in these three equations.

Define a, b and g by the equations $A = a\,a$, $B = b\,b$, $C = g\,c$ and then define *P*, *Q*, *R* by the equations

$$(3) \qquad \begin{aligned} P &= \frac{\sin p}{A} = \frac{\sin p}{a\,a} \\ Q &= \frac{\sin q}{B} = \frac{\sin q}{b\,b} \\ R &= \frac{\sin r}{C} = \frac{\sin r}{g\,c} \end{aligned}$$

Then the three formulas in (2) transform into

$$g R = P + Q,\ a P = Q + R,\ b Q = R + P$$

This gives us the ratios

$$P : R = g + 1 : a + 1$$
$$Q : P = a + 1 : b + 1$$
$$R : P = b + 1 : g + 1$$

This has a nice pattern, too. From this we get the triple proportion

$$(4) \qquad P : Q : R = \frac{1}{a+1} : \frac{1}{b+1} : \frac{1}{g+1}$$

though, to the modern eye, that doesn't look like a very convenient way to write anything.

Now, the first of our three equations in (3) gives

$$R = \frac{P+Q}{g}.$$

From the second equation we get $R = aP - Q$. Put these two values into the ratio between P and Q given in (4) and we get

$$\frac{P}{Q} = \frac{g+1}{ag-1}.$$

Since also $\frac{P}{Q} = \frac{b+1}{a+1}$, all this multiplies out to give

$$abg = a + b + g + 2.$$

Substituting back the triangle measurements for the Greek letters gives the result of the theorem.

QED

Here is an example of how we might be misled by reading an old theorem with modern eyes. Now we think of this theorem and the corollaries that we will see below as properties of triangles. That's not what Euler had in mind, though. This theorem gives a necessary property that the lengths of the six given segments must satisfy in order for him to solve the problem of finding the triangle that gives rise to those six lengths. That is to say, he still wants to solve the following:

Problem: Given that the parts of a triangle are lengths *A, B, C, a, b, c* as described above, to construct the triangle.

Euler's solution is two full pages long, and it is extremely analytical and non-geometrical. He even commits the "heresy of Heron and Brahmagupta" and uses calculations that involve square roots of fourth powers to find areas. Orthodox geometers objected to such calculations because fourth powers had no "real" geometric interpretation.

At this point in the paper, though, Euler is being true to his 18th century context, and uses fourth powers in geometry without agonizing over interpretation.

Having given an clunky proof to a somewhat awkwardly worded theorem, and used it to give a solution to a problem that holds little interest today, Euler spots a gem, and writes "the following most elegant consequence can now be stated:"

Theorem: In an arbitrary triangle *ABC*, draw from each angle *A*, *B* and *C* to a point on its opposite side straight lines *Aa*, *Bb* and *Cc* so that the three segments intersect at some point *O*. Then the segments always have the property that

$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}=1.$$

Proof follows from the previous theorem:

As above, take $AO = a \cdot Oa,\ BO = b \cdot Ob,\ CO = g \cdot Oc$. The previous theorem gives that

$$abg = a + b + g + 2.$$

Add $ab + ag + bg + a + b + g + 1$ to both sides. Then the left hand side factors to give

$$(a+1)(b+1)(g+1),$$

and the right hand side can be written as

$$ab + ag + bg + 2(a + b + g) + 3.$$

This last "obviously" [Euler's word] resolves to give

$$(a+1)(b+1)+(a+1)(g+1)+(b+1)(g+1).$$

Now, dividing both sides by the product

$$(a+1)(b+1)(g+1)$$

gives us

$$1=\frac{a}{a+1}+\frac{b}{b+1}+\frac{g}{g+1}.$$

QED

Euler tells us that he's done with the proof even though this last formula is only equivalent to what we were trying to prove, after substituting geometric segments for the Greek letters.

This theorem is the starting point for a recent article by Grünbaum and Klamkin. [G+K] They show, among other things, that the analogous sum of ratios also holds for tetrahedral, as well as for higher dimensional simplices. They correctly credit Euler for this result in two dimensions, but they neglect to notice that Euler also proved the two dimensional version of their Theorem 1 (ii). Euler doesn't put this particular result in the form of a theorem, but instead writes:

"Here is how the following memorable property can be derived:

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{g}{g+1}=2.$$

If this is added to the previous equation, it gives the following identity:

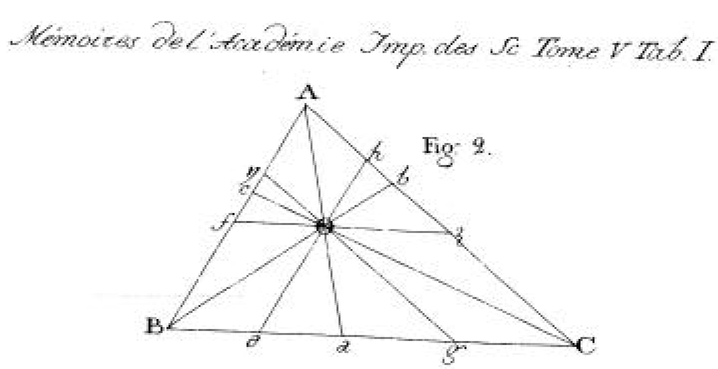
1+1+1 = 3."

Perhaps we should name this last equation the "Euler identity"? [1]

Note that Euler's three results so far, $\frac{a}{a+1}+\frac{b}{b+1}+\frac{g}{g+1}=2$, $1=\frac{a}{a+1}+\frac{b}{b+1}+\frac{g}{g+1}$ and $abg=a+b+g+2$ are algebraically equivalent. If we have a proof of any one of them, then the other two follow with just a little bit of algebra.

This seems like it would have been a fine place to stop this paper. Perhaps Euler even did stop here, because he continues with a proof of this last result that is so much nicer than the one he gave above that it seems like, if he'd known it when he was writing the earlier part of the paper, he would have used this proof instead. Or perhaps he came back to the paper later and added this part in. Regardless, he next gives us what he calls a "Most simple proof, based on ordinary elements." By this he means that he is going to use mostly geometry instead of mostly algebra to prove

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{g}{g+1}=1$$



Euler uses a new figure (Fig. 2), and some new notation, and some new lines in his figure.

Through O, he draws segments parallel to each of the three sides of ΔABC. The segment fz is parallel to BC, gh is parallel to AC and hq is parallel to AB.

[1] Joe Gallian take note. Do you think maybe the Beatles got their lyric "one and one and one is three" from this Euler paper?

With this notation, Euler writes the property he plans to prove in the form

$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}=1.$$

He begins his proof writing that since $AB = Ah + hf + fB$, we have

$$\frac{Bf}{AB}+\frac{Ah}{AB}+\frac{fh}{AB}=1. \tag{5}$$

Now, since ΔABa is similar to ΔAfO, we get $Bf : BA = Oa : Aa$, or, as fractions,

$$\frac{Oa}{Aa}=\frac{Bf}{BA}. \tag{6}$$

Save this to substitute into (5).

Likewise from $\Delta BAb \sim \Delta BhO$ we get another ratio that makes

$$\frac{Ah}{AB}=\frac{Ob}{Bb}. \tag{7}$$

The third of the similar triangles gives, in the same way $\frac{fh}{AB}=\frac{fO}{BC}$.

Now, by parallel lines, $fO = Bq$ and also $\Delta BCc \sim \Delta qCO$ so we get $\frac{Bq}{BC}=\frac{Oc}{Cc}$. That makes

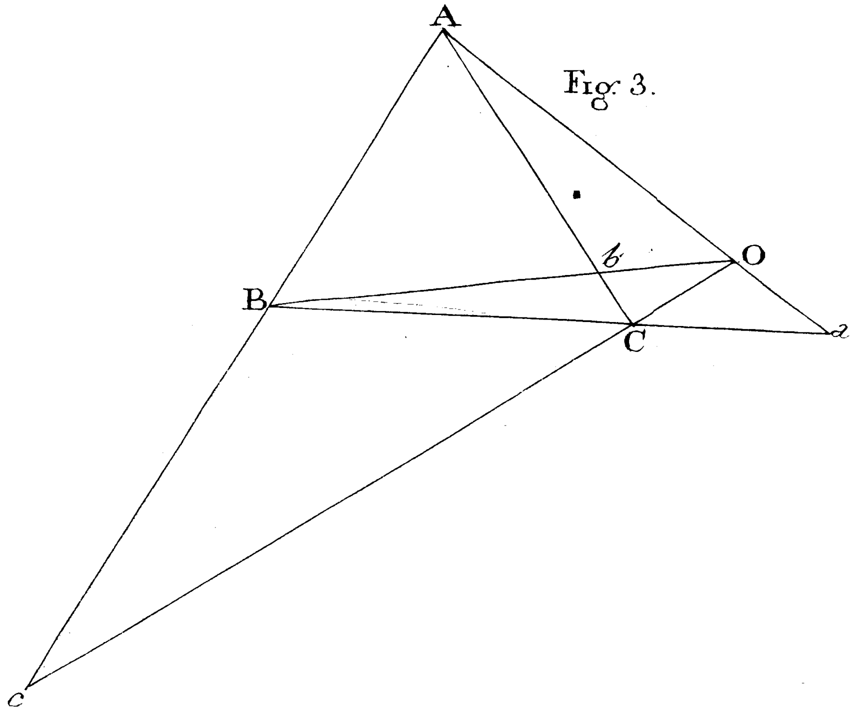
$$\frac{fh}{BA}=\frac{Oc}{Cc}. \tag{8}$$

Now we substitute formulas (6), (7) and (8) into the identity (5) $\frac{Bf}{AB}+\frac{Ah}{AB}+\frac{fh}{AB}=1$ and it gives our theorem

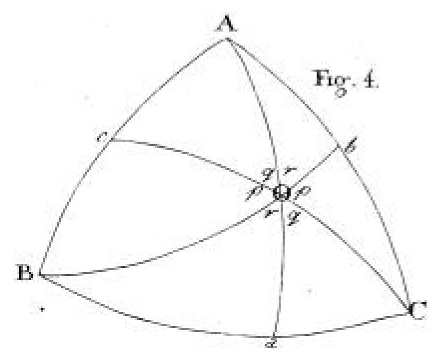
$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}=1.$$

QED

Euler adds that this property even holds if the point O is taken to be outside the triangle, as shown in Figure 3. Homer White, in footnotes to his translation of E749 (available on The Euler Archive) describes Euler's explanation of this property as "quite unclear." White also observes that the version of Figure 3 given in the *Opera Omnia* contains an error, reversing the labels on the points C and c. As we can see in our version of Figure 3, the points were correctly labeled in the original.



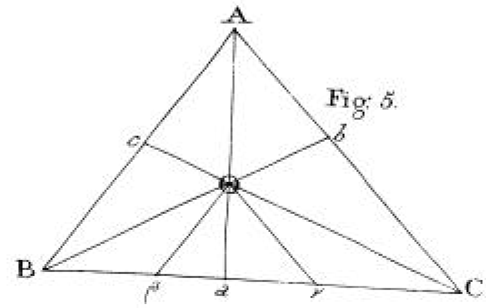
In typical Eulerian fashion, Euler proves that similar relations hold on coincident segments drawn across spherical triangles, as shown in Figure 4. He also shows how to solve his Problem for spherical triangles. These calculations closely resemble the calculations we saw early in the paper, reinforcing the hypothesis that the "most simple proof based on ordinary elements" may have been added later.



Whether or not that part was interpolated, the last part of this paper was clearly added later than the rest of the paper. Euler labels it "SUPPLEMENT Containing the simplest analysis for the proof of the theorem and for the solution of the problem proposed before."

Euler means to prove his theorem in the form $\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}=1$

For his proof, Euler uses Fig. 5, a simpler version of Fig. 2. Rather than add line segments parallel to all three sides, he adds only two shorter line segments, both from point O to points on side BC. Segment Ob is parallel to side AB and segment Og is parallel to AC. Obviously,



$$Bb + bg + gC = BC,$$

so

$$\frac{Bb}{BC} + \frac{bg}{BC} + \frac{gC}{BC} = 1 \qquad (9)$$

Now we use three pairs of similar triangles. $\Delta BCb \sim \Delta BgO$ so $\frac{Cg}{BC} = \frac{Ob}{Bb}$. Likewise $\Delta CBc \sim \Delta CbO$ so $\frac{Bb}{BC} = \frac{Oc}{Cc}$. Finally, $\Delta bOg \sim \Delta \mathrm{BAC}$, so $\frac{bg}{BC} = \frac{Oa}{Aa}$. Substitute these three fractions into (9), and immediately we get

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

QED

Euler is clearly proud of this proof and writes that "this is, without a doubt, the shortest proof of this theorem, but it was dug up in a most roundabout way."

He wraps up his paper with a not-quite-as-brief solution to his problem of finding the triangle given the segments. We note that, for this part, the illustration in the *Opera Omnia* again contains an error that was not present in the original.

So we reach the end of Euler's last published paper in Euclidean geometry. Because Euler didn't have the extra time to revise and polish this paper (he died just three years after writing it, and his SUPPLEMENT may have been added substantially closer to his death) he didn't "erase his tracks." Thus it gives us a glimpse of how Euler discovered things as he wrote a paper and how he came back later to improve his solutions.

It also shows how Euler, though nicknamed by one of his contemporaries *Analysis incarnate*, still had a flair for ordinary geometry and, though blind himself, still had an eye for a beautiful proof.

References:

[C+G] Coxeter, H. S. M., and S. L. Greitzer, *Geometry Revisited*, Aneli Lax New Mathematical Library, MAA, Washington DC, 1967. (Originally published by Random House, New York, 1967)

[E749] Euler, Leonhard, Geometrica et sphaericaquaedam, *Mémoires de l'Académie des sciences de St-Pétersbourg* **5** (1812) 1815, p. 96-114 reprinted in *Opera Omnia* Series I vol 26 p. 344-358. Available in its Latin original and a translation by Homer White through The Euler Archive at www.EulerArchive.org.

[G+K] Grünbaum, Branko and Murray S. Klamkin, Euler's Ratio-Sum Theorem and Generalizations, *Mathematics Magazine* **79**:2 (April 2006) 122-130.

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