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How Euler Did It

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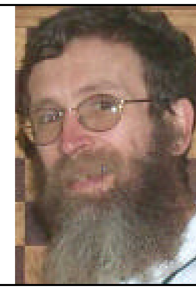

Infinitely many primes

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Why are there so very many prime numbers?

Euclid wondered this more than 200 years ago, and his proof that "Prime numbers are more than any assigned multitude of prime numbers" (Elements IX.20) is often considered one of the most beautiful proofs in all of mathematics.[1] Prime numbers, their detection, their frequency and their special properties remain at the heart of many of today's most exciting open questions in number theory.

This month, we return to one of Euler's early papers, *Variae observationes circa series infinitas*, to see what Euler has to say there about prime numbers. This paper has Eneström number 72, and we have already seen this paper in our column from February 2005, called "Goldbach's series." Euler wrote the paper in 1737, and it was published in 1744. Here we will find that Euler gives three answers to the question "How many prime numbers are there?" and, in a way, helps us understand why there are so many of them.

Since we discussed the first part of this paper last February, we will start in the middle of this paper, with Euler's Theorem 7:

> "The product continued to infinity of this fraction $\frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \text{etc.}}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot 12\cdot 16\cdot 18\cdot \text{etc.}}$ in which the numerators are prime numbers and the denominators are one less than the numerators, equals the sum of this infinite series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$, and they are both infinite."

The factors of the infinite product can be rewritten as $\frac{p}{p-1}=\frac{1}{1-1/p}$, so, in modern notation, this theorem can be rewritten as $\sum_{k=1}^{\infty}\frac{1}{k}=\prod_{p}\frac{1}{1-1/p}$. Some readers will recognize this as the sum-product formula for the Riemann Zeta function at the value $s = 1$. Others may recognize

[1] It was voted number 3 in a 1988 survey by *Mathematical Intelligencer*. See [W, p. 126]

it as part of the cover illustration of William Dunham's wonderful book on Euler and his work. [D] Euler offers the following proof of this theorem.

Proof: Let

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

Then

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.}$$

This leaves

$$\frac{1}{2}x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

Note that there are no even numbers left in the denominators on the right hand side. Now, to eliminate the denominators that are divisible by three, we divide both sides by three, to get

$$\frac{1}{2}\cdot\frac{1}{3}x = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \text{etc.}$$

Subtracting again eliminates all remaining denominators that are multiples of 3, leaving

$$\frac{1}{2}\cdot\frac{2}{3}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

This process is like the ancient sieve of Eratosthenes because, at each stage, it eliminates a prime denominator and all remaining multiples of that prime denominator. Eventually, everything on the right will be eliminated except the first term, 1. Euler carries us through one more iteration of his process. He divides this last equation by 5, and does a small rearrangement of the way he writes the product of fractions, to get

$$\frac{1\cdot 2}{2\cdot 3}\cdot\frac{1}{5}x = \frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \text{etc.}$$

Subtracting leaves

$$\frac{1\cdot 2\cdot 4}{2\cdot 3\cdot 5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

In the same way, terms with denominators that are multiples of 7, 11, and so forth for all prime numbers, will be eliminated, leaving

$$\frac{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot 12\cdot 16\cdot 18\cdot 22\cdot \text{etc.}}{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 21\cdot \text{etc.}}x = 1.$$

But, since x is already known to be the sum of the harmonic series, the desired result is immediate.

Q. E. D.

From its very first step, this proof does not satisfy modern standards of rigor, but it isn't too much work for a modern mathematician to recast the theorem to say that the harmonic series has a finite limit if and only if the infinite product does, and then to begin the proof "suppose that the harmonic series converges to a value x.

Nonetheless, if we accept the result, then we have a short proof that there are infinitely many primes. For the product $\frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \text{etc.}}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot 12\cdot 16\cdot 18\cdot \text{etc.}}$ to diverge it must be an *infinite* product, hence there must be infinitely many prime numbers.

Though it isn't exactly relevant to our topic, the next theorem, Theorem 8 is extremely important:

> Theorem 8: If we use the series of prime numbers to form the expression
>
> $$\frac{2^n}{(2^n-1)}\cdot\frac{3^n}{(3^n-1)}\cdot\frac{5^n}{(5^n-1)}\cdot\frac{7^n}{(7^n-1)}\cdot\frac{11^n}{(11^n-1)}\cdot \text{etc.}$$
>
> then its value is equal to the sum of this series
>
> $$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\text{etc.}$$

In modern notation, this is the familiar sum-product formula for the Riemann zeta function:

$$z\,(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=\prod_{p\,\text{prime}}\frac{1}{1-1/p^s},$$

and Euler proof of Theorem 8 is almost exactly like the proof of Theorem 7, with the exponent n included. Also, the proof of Theorem 8 is correct by modern standards because all the series involved are absolutely convergent.

Occasionally, someone will insist that, because Euler proved this Theorem 8, the function we call the Riemann zeta function ought to be called the *Euler* zeta function. We disagree. Euler's zeta function is a function of a real variable, and Euler never treats its value except for positive integers. Riemann extended the function to *complex* values, where its most interesting and important properties are found. If you want to, you can try to say the Euler zeta function is a function of a real variable, but the Riemann zeta function is complex. The forces of history probably won't let you though, and that is fair.

We jump forward to Theorem 19 for Euler's second result about how many prime numbers there are. His theorem is:

Theorem 19: The sum of the series of reciprocals of prime numbers

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\text{etc.}$$

is infinitely large, and is infinitely less than the harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.}$$

Moreover, the first sum is almost the logarithm of the second sum.[2]

Euler's proof of this again on adding and subtracting series that do not converge, and so does not meet modern standards of rigor. Unfortunately, Euler's proof can't really be "repaired." This time, we omit the proof.

Note that the first statement in the proof, that the series of reciprocals of primes diverges, not only tells us that there are infinitely many primes (since a finite series cannot diverge), but tells us that the primes are "dense" enough that the sum of their reciprocals diverges. The sum of the reciprocals of the square numbers, on the other hand, converges to $\frac{p^2}{6}$, as Euler showed a few years earlier when he solved the Basel problem. In this sense, there are "more" prime numbers than there are square numbers.

Finally, we come to Euler's third measure of how many primes there are. This comes from the second part of Theorem 19. This will require some interpretation and some analysis, but we hope to show that this remark is closely related to the so-called Prime Number Theorem.

Earlier in this paper, Euler made a cryptic remark about the "value" of the harmonic series, "if the absolute infinity is taken to be = ∞ then this expression will have the a value =ln ∞, which is the smallest of all the infinite powers."

The part about "=ln ∞" is fairly easy to interpret. In modern terms, it probably means that, for large values of n,

$$\sum_{k=1}^{n}\frac{1}{k}\approx \ln n\,.$$

The part about "smallest of all the infinite powers" isn't as clear, but I think it is about what we now call p-series, that the smallest value of p for which the p-series

$$\sum_{k=1}^{\infty}\frac{1}{k^p}$$

diverges is $p = -1$, the case of the harmonic series. That is, the harmonic series is the smallest power that gives an infinity.

[2] In Euler's Latin: *Atque illius summa est huius summae quasi logarithmus.*

Interpreting the rest of Euler's claim in the same way, it would seem to translate into the approximation

$$\sum_{\substack{p \text{ prime} \\ p<n}} \frac{1}{p} \approx \ln(\ln n).$$

We can find this as a theorem in modern number theory books, for example theorem 427 in [H+W].

We can differentiate this to find out how much the sum is expected to increase if we proceed from n to $n + 1$. We find that the derivative is $\frac{1}{n\ln(n)}$. We can apply some ideas from probability. If n is prime, then the sum will increase by $\frac{1}{n}$, and if n is not prime, then the sum will not increase. Hence, the "probability" that n is prime is about $\frac{1}{\ln n}$.

According to MacTutor [McT] "The statement that the density of primes is $1/\log n$ is known as the *Prime Number Theorem*." Moreover, Legendre observed this fact about the density of primes in 1798, and Gauss claimed to have observed it in 1793, but it was not proved until 1896 when Hadamard and de la Vallée Poussin independently discovered proofs. Precedence is usually given to Gauss's observation.

However, as we have just seen, the Prime Number Theorem is an easy consequence of Euler's Theorem 19.

Euler scooped Gauss by more than fifty years.

References:

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[H+W] Hardy, G. H, and E. M. Wright, *An Introduction to the Theory of Numbers*, 5ed, Oxford Univ. Press, 1979.

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[W] Wells, David, *The Penguin Book of Curious and Interesting Mathematics*, Penguin, London, 1997.

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