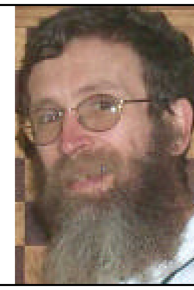
How Euler Did It

by Ed Sandifer



Who proved *e* is irrational?

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Most readers will know that the constant *e* is, indeed irrational, even transcendental. I remember being asked to prove *e* was irrational on my written exams for my master's degree. It is natural, then, to ask who was the first to prove it, and to expect an easy and unambiguous answer. The answer, though, isn't as easy as we might expect, nor is it entirely unambiguous.

Here is some of what MacTutor [McT] has to say about it:

> "Most people accept Euler as the first to prove that *e* is irrational. Certainly it was Hermite who proved that *e* is not an algebraic number in 1873."

Note that MacTutor hedges their attribution a bit. They write "*Most* people accept Euler as the first …," (my italics) and do not commit themselves to the more definite "Euler was the first …" In this case, Euler's rival is not some earlier mathematician who might have a claim to the result, but Euler's younger protégé Johann Heinrich Lambert (1728-1777), pictured at the right. Of Lambert, MacTutor writes:

> "Lambert is best known, however, for his work on π. Euler had already established in 1737 that e and e^2 are both irrational. However Lambert was the first to give a rigorous proof that π is irrational. In a paper presented to the Berlin Academy in 1768 Lambert showed that, if x is a nonzero rational number, then neither e^x nor tan x can be rational."

Note that MacTutor chooses words carefully, Euler "established," not "proved." On the other hand, Lambert's proof satisfies most standards of rigor. The hedging must be because people doubt the rigor of Euler's proof.

Our purpose in this month's column is to look at what Euler did, and to see just how rigorous Euler's results were.

Euler and Lambert both used the tools of continued fractions to produce their results. Euler's 1737 article that MacTutor mentions is "De fractionibus continuis dissertation" [E71]. Though Euler was not the first one to study continued fractions, this article is the first comprehensive account of their properties. Euler repeats most of the elementary properties of continued fractions in the last chapter of volume 1 of his 1748 masterpiece *Introductio in analysin infinitorum* [E101]. Both of these are available in excellent English translations.

The most general form of a continued fraction is

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

All the symbols, both Latin and Greek, are taken to be positive whole numbers. The Greek letters Euler calls *numerators* and the Latin are the *denominators*. In practice, the most interesting continued fractions are those for which all the numerators are 1. Continued fractions in this form are sometimes called *regular*.

It is not difficult to show that the regular continued fraction expansion of any rational number is finite, so to prove that a given number is irrational, it suffices to show that its regular expansion is not finite. We will show how this works using one of Euler's examples from the *Introductio*. We consider the number $\frac{e-1}{2} \approx 0.8591409142295 = \frac{8591409142295}{10000000000000}$.

Since this number is less than 1, the first denominator, $a = 0$. Now, Euler inverts the fractional part and gets

$$\frac{10000000000000}{8591409142295} = 1 + \frac{1408590847704}{8591409142295}.$$

The next denominator is the integer part of this, so $b = 1$. Invert the fraction part of this and get

$$\frac{8591409142295}{1408590847704} = 6 + \frac{139862996071}{1408590847704}.$$

The next denominator is the integer part of this, so $c = 6$. Continue to take integer parts, and invert fractional parts, and we get

$$\frac{1408590847704}{139862996071} = 10 + \frac{9950896994}{139862996071},$$

so $d = 10$.

$$\frac{139862996071}{9950896994} = 14 + \frac{551438155}{9950896994},$$

so e = 14.

$$\frac{9950896994}{551438155} = 18 + \frac{25010204}{551438155},$$

so f = 18.

$$\frac{551438155}{25010204} = 22 + \frac{1213667}{25010204},$$

so g = 22.

Euler stops here, saying "If the value for e at the beginning had been more exact, then the sequence of quotients would have been 1, 6, 10, 14, 18, 22, 26, 34, …, which form the terms of an arithmetic progression. It follows that

$$\frac{e-1}{2} = 0 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \text{etc.}}}}}}}\text{”}$$

Note that Euler's "arithmetic progression" doesn't start with the first denominator, but starts with the 6, after which the denominators increase by 4.

Euler adds, somewhat disingenuously, "This result can be confirmed by infinitesimal calculus."

Since the sequence of denominators clearly increases, and never terminates, this is not a finite continued fraction. Thus, by the work Euler did earlier, its value cannot be rational. Since $\frac{e-1}{2}$ is not rational, e cannot be rational, either.

By similar means, Euler shows that

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \text{etc.}}}}}}}}}}$$

Unless Euler skipped something, the proof is done.

Alas, Euler did skip something, and he hid it in that comment, "This result can be confirmed by infinitesimal calculus." He has only *observed* that finite calculations lead to a pattern for the first few denominators, and that the pattern seems to extend indefinitely. He has not *proved* it, and he knows he has not proved it.

What could he have been thinking?

Earlier in the chapter of the *Introductio*, Euler showed how to convert a continued fraction, whether regular or not, into an alternating series. He showed that if

$$x = a + \cfrac{a}{b + \cfrac{b}{c + \cfrac{g}{d + \cfrac{d}{e + \cfrac{e}{f + \text{etc.}}}}}}$$

then

$$x = a + \frac{a}{b} - \frac{ab}{b\left(bc+b\right)} + \frac{abg}{\left(bc+b\right)\left(bcd+bd+gb\right)} - \ldots$$

Perhaps we can apply this alternating series formula to the coefficients we got for $\frac{e-1}{2}$ and get something related to one of the well-known series for e? But, if we do that, we get

$$x = 1 - \frac{1}{7} + \frac{1}{7 \cdot 71} - \text{etc.}$$

which does not seem related to any other well-known series for e.

So, that wasn't how he did it. If we go back to E-71, we get more clues. In fact, he writes:

> "In the preceding sections, where I have converted the number e (whose logarithm is 1) together with its powers into continued fractions, I have only observed the arithmetic progression of the denominators and I have not been able to affirm anything except the probability of this progression continuing to infinity. Therefore, I have exerted myself in this above all: that I might inquire into the necessity of this progression and prove it rigorously. Even this goal I have pursued in a peculiar way."

Indeed his solution comes from a most surprising direction, differential equations. It lies in a form of an important differential equation called the *Ricatti equation*:

$$a\,dy + y^2 dx = x^{\frac{-4n}{2n+1}} dx$$

Euler claims that if we substitute $p=(2n+1)x^{\frac{1}{2n+1}}$, then the equation transforms into

$$a\,dq+q^2dp=dp$$

"I have found that

$$q=\frac{a}{p}+\cfrac{1}{\cfrac{3a}{p}+\cfrac{1}{\cfrac{5a}{p}+\cfrac{1}{\cfrac{7a}{p}+\ldots+\cfrac{1}{\cfrac{(2n-1)}{p}+\cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}}\text{"}$$

This continued fraction terminates after n ratios, but if n is taken to be one of Euler's "infinite numbers," then the continued fraction goes on forever.

This is a great leap. If Euler did show this previously, then I couldn't find where he did it. The most likely places to look would be in E-28 and E-31, where Euler does other series analyses of the Ricatti equation, but I can't see it there, or in any other papers on differential equations that Euler wrote before he wrote E-71. It was a great mystery to me.

On the other hand, the variables in the equation $a\,dq+q^2dp=dp$ separate to give

$$\frac{a\,dq}{1-q^2}=dp$$

which, in turn, integrates to give

$$\frac{a}{2}\log\frac{1+q}{1-q}=p+C\,.$$

The constant C can be taken to be zero with the initial conditions $q=\infty$ and p = 0. A bit of algebra gives that

$$e^{\frac{2p}{a}}=\frac{q+1}{q-1}=1+\frac{1}{q-1}$$

Just a little while ago, though, we got a continued fraction expansion for q, which we can substitute into this last expression to get

$$e^{\frac{2p}{a}}=1+\cfrac{1}{\cfrac{a-p}{p}+\cfrac{1}{\cfrac{3a}{p}+\cfrac{1}{\cfrac{5a}{p}+\cfrac{1}{\cfrac{7a}{p}+\text{etc.}}}}}$$

Various values of p and a give continued fractions for various expressions involving e. For example, p =1 and a = 2 gives

$$e = e^1 = e^{\frac{2\cdot 1}{2}}$$

$$= 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \text{etc.}}}}}$$

which is equivalent to the expansion for $\frac{e-1}{2}$ that Euler had observed earlier.

Euler works through a few other substitutions to derive his other observations, then writes,

> "Truly everything found above follows from these formula, by which we have expressed *e* and its powers as continued fractions. That is, the necessity of the progressions only observed earlier is now proved."

So, we complete the path from Euler's continued fraction solution to the Ricatti equation to the irrationality of *e*, but we can't be very satisfied with that solution of the differential equation. I looked at a good sample of Euler's earlier work, and can't find where Euler might have discovered this solution. I was about to give up and admit Euler's claim to having proved the irrationality of *e* had a great big hole in it.

So, I was about to throw in the towel and say Euler's claim to proving the irrationality of *e* was kind of weak. I wasn't quite ready to let go of it, when I had one of those "right under my nose" experiences. There it was in the last five paragraphs of E-71. Euler gives, in considerable detail, his proof that the continued fraction solves the Ricatti equation. We won't go into much detail; the interested reader can find the details in the Wyman and Wyman translation [E71], starting in paragraph 31. Briefly, he starts with a regular continued fraction in which the denominators form an arithmetic series. It looks like

$$s = a + \cfrac{1}{(1+n)a + \cfrac{1}{(1+2n)a + \cfrac{1}{(1+3n)a + \cfrac{1}{(1+4n)a + \text{etc.}}}}}$$

He uses his identities from early in the paper to rewrite this as a ratio of power series involving *n* and *a*, then shows that the power series in the numerator of the ratio is related to the derivative of the power series in the denominator. This gives him a differential equation, which, three pages later, he transforms into the Ricatti equation he wanted.

It's right. It's complete, and it works. I'd been fooled when Euler suggested that he had already shown the relation between the continued fraction and the differential equation. Euler really did prove that *e* is irrational, and he probably regarded it as the main point of this paper.

We're ready to close this month's column. There was probably a shorter path from the question "Who proved *e* is irrational?" to the conclusion "Euler," but this path shows some of the details of how we learned the story. We hope you've enjoyed the adventure.

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