

	<h1>How Euler Did It</h1> <p>by Ed Sandifer</p>	
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## $2aa + bb$

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How do we know what to try to prove?

A logician, or perhaps a Euclidean geometer, might try to say that we don't *try* to prove anything. We select some axioms or hypotheses. We apply some rules of inference and build a proof. Then the last line of the proof tells us what we've proved.

Uncharitable people sometimes claim that philosophers omit the axioms from this process, that politicians omit the rules of inference, and that people in the humanities never get to the last line,

A scientist, on the other hand, might claim that there is no need for axioms or rules of inference. One need only collect the data, and a correct analysis of the data will reveal the truth.

Much of the culture and image of mathematics is built on this "creation myth," that mathematical theorems are revealed in their statements, and that they are discovered by their proofs. Mathematical truths are imbued with a kind of crystalline purity, true in some absolute sense and unsullied by such vague and uncertain processes like experimentation and creativity.

Today, these notions may seem like idle, post-modernist speculations, but in Euler's time there was a great controversy in science over whether science should be based on observation or on deduction. In rough terms, the sides lined up as Newton vs. Leibniz. Their disagreements weren't based only on the priority dispute in calculus. Leibniz followed in the tradition of Descartes and believed that one should start with known truths and then apply logical methods to discover the truths that must follow from the known truths. Descartes had promoted this basis for reasoning in his *Method*, and used it to with great success to discover analytic geometry and to give the first correct explanation of the colors of the rainbow.

Newton, on the other hand, placed a great value on observations. He would make observations, then formulate theories that seemed to explain those observations. He would test those theories, and, if necessary, revise the theories. However, when he explained his theories, he, like Archimedes, would frequently hide the methods by which he made his discoveries.

Of course, Newton was not purely “Newtonian,” just as Leibniz and Descartes were not purely “Cartesian,” but these are the rough outlines of their disputes. Further details are in Hall’s fine book *Philosophers at War*. [H]

Euler practiced observation in his work on applied mathematics, though he often hid his method, in the style of Archimedes. He also followed Newton and Descartes in replacing the constructive methods of Geometry with the analytic methods of algebra and calculus.

Early in his career, Euler tended to be Leibnizian and Cartesian. As he matured, he selected principles from both sides of the dispute, but in general he became more and more Newtonian. His exposition, though, seemed more and more Leibnizian, as he developed a very modern-looking style of theorem – proof – corollary. One could think from his writing that he was a “proof machine” that never made an observation or made a conjecture.

In 1756, Euler decided to “come clean” about how he knew what to try to prove. He wrote a paper, *Specimen de usu obserationum in mathesi pura*, “Example of the use of observation in pure mathematics,” [E256] in which he describes his path from observation to theorem. He attributes the technique, on slim evidence, to Fermat.

To explain his method, Euler selects material from some of his then-recent papers on number theory, especially E 164 (the principal subject of last month’s column), about the quadratic forms  $aa + pbb$ , and E 241, in which he gives his proof that the prime numbers of the form  $4n + 1$  are exactly the ones that are the sum of two squares.

Here in E 256, Euler studies numbers of the form  $2aa + bb$ , a special case of the numbers he studied in E 164. After a two page introduction about the relation between observation and proof, Euler begins his work with eight observations about numbers of the form  $2aa + bb$ , taking  $a$  and  $b$  to be relatively prime. This takes only two pages. He spends the last 15 pages of the paper trying to prove these eight observations. As it turns out, he isn’t able to prove all of them, and the things that were hardest to observe aren’t always the hardest ones to prove. In the course of his proofs, though, Euler comes across other things that are true, and proves them, too.

Euler begins his observation with a list. He tabulates all the numbers less than 500 of the form  $2aa + bb$ , with  $a$  and  $b$  relatively prime. His list looks something like this:

$2 + bb$ )	3, 6, 11, 18, 27, 38, 51, 66, 83, 102, 123, 146, 171, 198, 227, 258, 291, 326, 363, 402, 443, 486.
$8 + bb$ )	9, 17, 33, 57, 89, 129, 177, 233, 297, 369, 449.
...	
$450 + bb$ )	451, 454, 466, 499.

Note that Euler expects both  $a$  and  $b$  to be non-zero. Now he starts mining his list for information and making his observations:

**Observation 1:** We look at the 45 prime numbers that appear on the list:

3, 11, 17, 19, 41, 43, 59, 67, 73, ... 491, 499.

None of these numbers appears more than once on the list, hence we speculate that such prime numbers are *uniquely* represented in this form. As Euler gets to proving the theorems behind these

observations, it becomes clear that this is really two statements, first that prime numbers appear only once, and second that (odd) numbers that appear only once are prime.

**Observation 2:** Next we list the doubles of the prime numbers.

6, 22, 34, 38, 82, 86, 118, 134, 146, ... 466, 482.

They, too, appear only once each, so they, too, are uniquely represented. They are exactly the doubles of the primes in the first list, and there are no numbers on the list that are multiples of 4.

Since before Euclid's time, about 2400 years ago, numbers that are divisible by 2 but not divisible by 4 have been called "oddly even." Observation 2 says, among other things, that even numbers of the form  $2aa + bb$  are oddly even. Knowing this, we quote Euler's next observation.

**Observation 3:** "Compare the numbers that are odd and the ones that are even, but oddly even, and I observe: *If an odd number is represented, then so also is its double, and also, if an even number appears, half of it will appear as well.*"

**Observation 4:** For those remaining numbers (i.e., not prime, also not even) list their prime factorization, and at the same time, in parentheses give the number of times each number appears in the list:

$3^2$  (1)  $3^3$  (1)  $3 \cdot 11$  (2),  $3 \cdot 17$  (2)  $3 \cdot 19$  (2),  $3^4$  (1),  $3^2 \cdot 11$  (2),  $11^2$  (1),  $3 \cdot 41$  (2), etc.

From this we see that any product of the prime numbers we saw in Observation 1 also occurs on the list, and it occurs more than once if it is composed of different factors. For example, 33 has two prime factors, 3 and 11, and it occurs twice on the list because it has two different representations of the form  $2aa + bb$ , being  $2 \cdot 4 + 25$  and  $2 \cdot 16 + 1$ .

Note that Euler does NOT claim, though it is true, that the number of times a number occurs doubles for each odd prime factor it has.

To deal with the special prime number 2, Euler specifically notes that we can get it by taking  $b = 0$  and  $a = 1$ , despite his general assumption that both  $a$  and  $b$  be non-zero.

**Observation 5:** Among the factors of these numbers there are no primes except those that are also of the form  $2aa + bb$ .

**Observation 6:** No prime numbers of the forms  $8n - 1$  or  $8n - 3$  are of the form  $2aa + bb$ , nor can they be divisors of numbers of the form  $2aa + bb$ , as long as  $a$  and  $b$  are relatively prime.

**Observation 7:** No number of the form  $2aa + bb$ , with  $a$  and  $b$  relatively prime has any prime divisors other than 2 and prime numbers of one or the other forms  $8n + 1$  or  $8n + 3$ .

**Observation 8:** Now it is of greatest interest to observe that every prime number of these two forms  $8n + 1$  and  $8n + 3$  occurs on the list.

Euler notes that all of these observations are easy to make, and some of them can be proved, but for others the proofs are "most difficult." Into the first category (the easy ones) fall observations 1, 2, 3, 4, and the first part of 6. The hard ones are 5, the second part of 6, and 7. The very deepest, he says, is

Observation 8. Moreover, he notes, these properties are similar in many ways to the properties of the sums of two squares that he described in E 228 and E 241.

Now that he has shown us how observations give him ideas about what to prove, the character of this paper changes dramatically. Euler sets out to prove the things he observed, and he uses his usual Theorem-Proof-Corollary structure, with a few examples put in near the end.

As we mentioned above, Euler does not prove the same things he observes, and when he proves these observations, he doesn't prove them in the same order he observed them, either. His first theorem is a proof of the first part of Observation 3. We quote:

**“Theorem 1:** If  $N$  is of the form  $2aa + bb$ , then so is its double.

Proof: Take  $N = 2mm + nn$ , so that  $2N = 4mm + 2nn$ . Take  $2m = k$ . This makes  $2N = kk + 2nn$ , and so  $2N$  is a number of the form  $2aa + bb$ . Q.E.D.”

Predictably, Theorem 2 is the converse of Theorem 1, and completes his proof of Observation 3, which, in turn, implies the result in Observation 2. Again we quote:

**“Theorem 2:** If a number  $2N$  is of the form  $2aa + bb$ , then so also its half,  $N$  is of the same form.

Proof: For this to happen, it is necessary that  $nn$  be even, and so  $n$  itself is even. Write  $n = 2k$ , so that  $2N = 2mm + 4kk$  and so  $N = mm + 2kk$ , which is a number of the form  $2aa + bb$ . Q.E.D.”

Something about this gave Euler the idea of asking whether the product of two numbers of the form  $2aa + bb$  was again a number of that form, even though he had not made any observations about the question. He answers the question in the affirmative, and adds a little bit, with Theorem 3. Euler's proof of his Theorem 3 is a bit longer and wordier, so we only paraphrase it:

**Theorem 3:** If  $M$  and  $N$  are of the form, then so is their product  $MN$ .

Proof: Take  $M = 2aa + bb$  and  $N = 2cc + dd$ . Then

$$\begin{aligned} MN &= 4aacc + 2aadd + 2ccbb + bbdd \\ &= 4aacc + 4acbd + bbdd + 2aadd - 4acbd + 2ccbb \\ &= (2ac + bd)^2 + 2(ad - cb)^2 \end{aligned}$$

Note also that if we reverse the signs of the added terms, we get  $MN$  in a second way, as

$$\begin{aligned} MN &= 4aacc + 2aadd + 2ccbb + bbdd \\ &= 4aacc - 4acbd + bbdd + 2aadd + 4acbd + 2ccbb \\ &= (2ac - bd)^2 + 2(ad + cb)^2 \end{aligned}$$

Moreover, these two ways must be different.

Note that there might be a little hang-up if any of the calculated values come out negative. If that happens, take their corresponding positive values. Q.E.D.

Surely Euler noticed that this same proof would work for any quadratic form  $2aa + bb$ , but he does not mention that here. He is doing a good job maintaining his focus on the particular form  $2aa + bb$ .

Now Euler turns to his first observation, that the prime numbers appear only once on the list. In his Theorem 4, he states the result in its contrapositive, and then proves it by contradiction. Although the proof is a bit long, it is a very clever proof. Also, Euler doesn't do such straightforward proofs by contradiction very often, so it is interesting for that as well. Again, we paraphrase:

**Theorem 4:** Any number that can be resolved in two ways into a form  $2aa + bb$  is not prime.

Proof: Euler uses proof by contradiction.

Suppose  $N$  is prime and  $N$  can be resolved in two different ways. Say  $N = 2aa + bb$  and  $N = 2cc + dd$ , with  $a$  and  $b$  different from  $c$  and  $d$ . Multiply the first resolution by  $cc$  and the second by  $aa$  and subtract to get, on the one hand,  $(aa - cc)N$ , and on the other hand  $aadd - bbcc$ , which factors as the difference of squares as  $(ad - bc)(ad + bc)$ .

Since  $N$  is prime, it must divide one or the other of these factors. This is the consequence that Euler will contradict.

Now also, add the two forms and get

$$2N = 2aa + bb + 2cc + dd.$$

From this take away  $2ad + 2bc$  and there remains

$$2N - 2ad - 2bc = 2aa + bb + 2cc + dd - 2ad - 2bc$$

which can be reorganized as

$$2N - 2ad - 2bc = aa + (a - d)^2 + cc + (c - b)^2.$$

Now, the RHS is the sum of four squares, and so it is certainly greater than zero, and so

$$2N - 2ad - 2bc > 0, \text{ so}$$

$$N > ad + bc$$

So  $N$  must also be greater than  $ad - bc$ , and so  $N$  can divide neither  $ad + bc$  nor  $ad - bc$ , so the consequence we flagged above cannot be true.

All this derived from the hypothesis that there were two resolutions, so there can't be two distinct resolutions of a prime number. **QED.**

Observation 5 is next on Euler's agenda, that all prime factors of a number of the form  $2aa + bb$  (where  $a$  and  $b$  are relatively prime) must also be of that form. This is the first of the observations that Euler described as being more difficult. Indeed, his Theorems 5 to 8 are rather difficult technical lemmas that lead to Theorem 9, which we quote:

**“Theorem 9:** No number of the form  $2aa + bb$ , for which  $a$  and  $b$  are relatively prime, can have a prime factor that is not also of this form.”

Theorems 10 and 11 explain Observation 1 and the first part of Observation 6, respectively:

**Theorem 10:** If a number of the form  $2aa + bb$  resolves into this form in just one way, and if  $a$  and  $b$  are relatively prime, then the number is certainly prime.

**Theorem 11:** No number of the forms  $8n - 1$  or  $8n - 3$  can divide any number  $2aa + bb$ , as long as  $a$  and  $b$  are relatively prime.

Euler's last theorem in this vein is related to the other results, but it is not explicit among his eight observations:

**Theorem 12:** If a number in one or the other of the forms  $8n + 1$  or  $8n + 3$  cannot be resolved into the form  $2aa + bb$ , then it is not prime; and if can be so resolved in exactly one form, then it is prime; and if it can be resolved in more than one such way, then it is not prime, but it is composite.

Except for the part about observations at the beginning, this paper really has turned into a fairly typical Euler paper in number theory. In true Eulerian form, there are two examples.

First, Euler asks whether the number 67579 is prime. He sees that it is of the form  $8n + 3$ , and so, by Theorem 12, he can show it is prime by showing that it is uniquely of the form  $2aa + bb$ . He does this more or less by brute force, and finds that  $67579 = 2 \cdot 87^2 + 229^2$ , and this is its only representation in the form  $2aa + bb$ , hence it is prime.

In Euler's time, 67579 is not a particularly large prime number. They knew several seven-digit prime numbers, but didn't know any eight-digit ones yet.

Euler's second example is to demonstrate that 40081 is not a prime number. Though it is of the form  $8n + 1$ , it is not of the form  $2aa + bb$ , so, again by Theorem 12, it is not prime. In fact, it is the product of 149 and 269, but nothing in this technique helps to find the factorization.

If Euler had ended the paper here, the bulk of this paper would be much like many of his other papers, but he has a surprise ending, two entirely unexpected theorems about square and triangular numbers that are corollaries of the theorems he has already proved:

**Theorem 13:** If a number  $n$  is in no way the sum of a square and a triangular number, then the number  $8n + 1$  certainly is not prime.

Proof: If  $n$  were not of the form  $aa + \frac{1}{2}(bb + b)$ , then  $8n + 1$  could not be of the form

$$8aa + 4bb + 4b + 1,$$

and hence could not be (taking  $p = 2a$  and  $q = 2b + 1$ ) of the form  $2pp + qq$ , and so could not be prime. Q. E. D

This is a negative result. Its contrapositive, that if  $8n + 1$  is prime, then  $n$  is a square plus a triangle, is only a necessary, and not a sufficient condition. For (my) example,  $10 = 9 + 1$  is a square and a triangle, but 81 is not prime.

**Theorem 14:** If  $n$  is in no way the sum of a triangular number and the double of a triangular number, then  $8n + 3$  is certainly not prime.

Proof: Here our number does not start as  $aa + a + \frac{1}{2}(bb + b)$ , which, multiplying by 8 and adding 3 (in the form  $2 + 1$ ) gives  $8aa + 8a + 2 + 4bb + 4b + 1$ , which is of the form  $2pp + qq$ . Hence, our number is *not* of this form, and cannot be prime. Q. E. D.

The Reader is encouraged to verify these surprising results with some further experiments.

This ends Euler's remarkable description of the delicate dance between observation and deductive proof, complete with examples and a surprise ending.

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