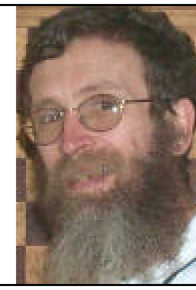
How Euler Did It

by Ed Sandifer



Amicable numbers

November 2005

Six is a special number. It is divisible by 1, 2 and 3, and, in what at first looks like a strange coincidence, 6 = 1 + 2 + 3. The number 28 shares this remarkable property; its divisors, 1, 2, 4, 7 and 14, sum to the number 28. Numbers with this property, that they are the sum of their divisors (including 1, but not including the number itself) have been known since ancient times and are called *perfect numbers*. Euclid himself proved in Book IX, proposition 36 of the *Elements* [E]:

> If as many numbers as we please beginning from an unit be set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last make some number, then the product will be perfect.

In a more modern treatment, Hardy and Wright [H+W] state this same theorem as

> THEOREM 276: If $2^{n+1}-1$ is prime, then $2^n\left(2^{n+1}-1\right)$ is perfect.

Each such perfect number is associated with a prime of the form $2^{n+1}-1$, and such numbers are now called *Mersenne primes*. Several Mersenne primes are known, and for several decades, the largest known prime number was usually a Mersenne prime. This is no longer the case.

Euler proved that all even perfect numbers have the form in Theorem 276, and also discovered a few properties that a perfect number would have to have if it were odd. Since no odd perfect numbers are known, it is difficult to explain to non-mathematicians why it might be interesting to prove things about them anyway. As far as I know, the two best-known properties of odd perfect numbers are:

1. There might not be any, and
2. if there are any, they must be very large.

But we are off the track of the story. Consider the pair of numbers, 220 and 284. The divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, and those divisors sum to 284. Meanwhile, the divisors if 284 are 1, 2, 4, 71 and 142, and they sum to 220. Such pairs of numbers, the divisors of one summing to the other, are called *amicable pairs*.

For over a thousand years, only this pair, 220 and 284, was known. Iamblichus, in the fourth century BCE, wrote, “The first two friendly numbers are these: sigma pi delta and sigma kappa.” In the

Greek number system in use at the time, sigma had a value 200, pi and kappa were 80 and 20 respectively, and delta was 4, so he was describing 284 and 220.

In the 9th century, Arab mathematician Thabit ibn Qurra probably discovered the next amicable pair, 17296, 18416. In the 1600's, Pierre Fermat rediscovered this pair, and his mathematical rival René Descartes discovered another pair, 9,363,584 and 9,437,056.

So, when Euler came on the scene, only three pairs of amicable numbers were known. Then, in 1747, Euler published a short paper [E100] mentioning the technique that Descartes and Fermat had used, and listing 30 amicable pairs, including the three already known, and including one "pair" that was not actually amicable. Nevertheless, in one paper, Euler lengthened the list of known amicable pairs by a factor of almost ten.

Euler gives us almost no clue about how he found these numbers. He briefly describes the methods Descartes and Fermat had used, though. They had considered pairs of numbers of the form $2^n xy$ and $2^n z$, where *x*, *y* and *z* are all prime, and showed that, for the numbers to be an amicable pair, it was necessary that $z = xy + x + y$. Fermat and Descartes had just searched for prime numbers *x*, *y* and *z* to see which ones gave amicable pairs.

However, this cannot be how Euler found his new amicable pairs, since only the first three, the ones that were already known, have this form. Eleven of the others have the form $2^n xy$ and $2^n zw$, where *x*, *y*, *z* and *w* are all prime, but others involve as many as seven distinct prime factors, and ten of the pairs are pairs of odd numbers.

It is not like Euler to leave us in the dark like this, without showing us how he made his discoveries, and I can offer no very satisfying explanation. It is true that most articles published in the *Nova acta eruditorum* were rather brief, but this article was only three pages long. An author of Euler's stature would have been welcome to write six or seven pages, if he had wanted to. It is also true that few important mathematicians had worked on number theory since the days of Fermat, who died in 1665, 80 years before Euler wrote this article, and this was only Euler's sixth article that the Editors of the *Opera Omnia* classify as "number theory." Since Euler published over 90 such articles, E100 comes quite early in his number theory career. Neither of these seems to explain why Euler chose to be so obscure.

Later in 1747, though, Euler wrote another paper, *Theoremata circa divisores numerorum*, or "Theorems about divisors of numbers," [E134] in which he explained how he had discovered that the fifth Fermat number, $2^{2^5}+1$ was not prime, but was divisible by 641, and also gave his first proof of Fermat's Little Theorem. This was the subject of the very first column in this series, back in November of 2003. Perhaps that paper got Euler thinking about providing better explanations of his discoveries in number theory, or maybe it just kept him interested in number theory.

Whatever the reason, in 1750, Euler returned to the problem of amicable numbers, armed with a powerful new idea, the first of what we now call *number theoretical functions*. He invented a new function and a new notation, denoting the sum of the divisors of a number *n*, including *n* itself, by $\int n$. The integral sign is supposed to remind us that we are summing something. This function is now sometimes called the *sigma function* and denoted $s(n)$. Here we will use Euler's notation.

Immediately after introducing his new notation, Euler gives the example that $\int 6 = 1 + 2 + 3 + 6 = 12$, and that, in general, perfect numbers are those for which $n = \int n - n$ and prime numbers are those for which $\int n = 1 + n$. He pays due attention to his fundamental case, $\int 1 = 1$, and notes that this shows that "the unit ought not be listed among the prime numbers."

He follows with an exposition almost indistinguishable from that in a modern number theory textbook, of the basic properties of his new function:

Lemma 1: If *m* and *n* are relatively prime, then $\int nm = \int m \cdot \int n$

Corollary: If *m, n* and *p* are prime numbers, the

$$\text{n}\int mnp = \int m \cdot \int n \cdot \int p = (1+m)(1+n)(1+p)$$

Lemma 2: If *n* is a prime number, then $\int n^k = 1 + n + n^2 + \ldots + n^k = \frac{n^{k+1} - 1}{n-1}$.

Lemma 3: If a number *N* has a prime factorization $N = m^a \cdot n^b \cdot p^g \cdot q^d \cdot \text{etc.}$, then

$$\int N = \int m^a \cdot \int n^b \cdot \int p^g \cdot \int q^d \cdot \text{etc.}$$

Euler does a few examples like finding that $\int 360 = 1170$ and using his new function to show that 2620 and 2924 form an amicable pair. With this last example, he is showing off a bit, since this pair is not among the three amicable pairs known in ancient times, though it was on his list in E100. Then he turns to characterizing amicable numbers. Here is how he does it.

If *m* and *n* are amicable pairs, then $\int m - m = n$ and $\int n - n = m$, and a little bit of algebra leads to the form Euler wants: $\int m = \int n = m + n$. Armed with this, he begins to study amicable pairs that share a common factor, *a*. He classifies these as follows:

$$\text{first form} \begin{cases} apq \\ ar \end{cases} \qquad \text{second form} \begin{cases} apq \\ ars \end{cases}$$

$$\text{third form} \begin{cases} apqr \\ as \end{cases} \qquad \text{fourth form} \begin{cases} apqr \\ ast \end{cases} \qquad \text{fifth form} \begin{cases} apqr \\ astu \end{cases}$$

A modern reader might want to count the factors that the pairs do not have in common, and then classify these with a notation like (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), etc. We could then try to make a case that it resembles Cantor's diagonal proof that the rational numbers are countable, but such observations are anachronistic, and are more amusing than they are useful or valid.

Now he considers these one form at a time.

PROBLEM 1

First, Euler considers amicable pairs of the form *apq* and *ar,* where there is a common factor *a* and the numbers *p, q* and *r* are prime numbers and not factors of *a*. All of the amicable pairs known before Euler's time were of this form and had *a* being a power of 2.

The condition he found earlier implies that $\int r = \int p \cdot \int q$, and, since *p, q* and *r* are prime, this means that

$$r + 1 = (p + 1)(q + 1).$$

Substituting *x* for *p* + 1 and *y* for *q* + 1, this makes *r* = *xy* – 1, where the numbers *x* – 1, *y* – 1 and *xy* – 1 must all be prime, and the numbers *a*(*x* – 1)(*y* – 1) and *a*(*xy* – 1) form the amicable pair he is seeking. Moreover, the condition $\int m = \int n = m + n$ becomes

$$a\left(2xy-x-y\right)=xy\int a \text{ or } y=\frac{ax}{\left(2a-\int a\right)x-a}.$$

He simplifies this with the substitution $\frac{b}{c}=\frac{a}{2a-\int a}$, $\frac{b}{c}$ taken to be in lowest terms.

Substituting this into the expression for *y*, he gets the fairly simple form

$$\left(cx-b\right)\left(cy-b\right)=bb$$

and, because *p, q* and *r* are prime, he gives the additional conditions that $x-1$, $y-1$ and $xy-1$ must all be prime.

This is enough new information to start searching for amicable pairs. He begins what he calls Rule 1, and supposes that *a* is a power of 2, say $a=2^k$. His substitutions lead to $b=2^n$ and $c=1$, so that

$$\left(x-2^n\right)\left(y-2^n\right)=2^{2n}.$$

Euler didn't leave out any steps of this calculation, but in Euler's day, paper was expensive, and we have a choice of cheap paper or computer algebra systems if we want to check his work.

Continuing, there aren't very many ways to factor 2^{2n}, and this product must have the form

$$\left(x-2^n\right)\left(x+2^n\right)=2^{n+k}\cdot 2^{n-k}$$

for some value of *k*. From this it follows that

$$x=2^{n+k}+2^n$$

$$y=2^{n-k}+2^n$$

and the three prime numbers *p, q* and *r* that to in to making the amicable pair are

$$p=x-1=2^{n+k}+2^n-1$$

$$q=y-1=2^{n-k}+2^n-1$$

$$r=xy-1=2^{2n+1}+2^{2n+k}+2^{2n-k}-1$$

Euler makes one more, in this case rather unnecessary substitution, taking $m=n-k$, so that $n=m+k$, and rewrites these equations in terms of *m* and *k* instead of *n* and *k*. We'll skip that.

Now, Euler considers as separate cases various values of *k*.

First, if $k=1$, we look for primes of the forms

$$p=3\cdot 2^m-1$$

$$q=6\cdot 2^m-1$$

$$r=18\cdot 2^{2m}-1$$

If $m=1$, then these give prime numbers 5, 11 and 71, and so the numbers

$$220=2^2\cdot 5\cdot 11$$

$$284=2^2\cdot 71$$

is an amicable pair.

If $m=2$, we get the numbers 11, 23 and 287. The first two are prime, but the third is 7×41, and so this case does not yield an amicable pair.

If $m=3$, we get the numbers primes 23, 47 and 1151, and hence the amicable pair

$$17{,}296 = 2^4 \cdot 23 \cdot 47$$
$$18{,}416 = 2^4 \cdot 1151$$

After a few more unfruitful substitutions for $m = 4$ and $m = 5$, taking $m = 6$ gives another amicable pair, which we will leave to the reader to calculate.

Euler also considers cases $k = 2$, 3, 4 and 5, but they yield no additional amicable pairs. He assures us that these three are the only amicable pairs of this first form involving a common factor of 2^n and involving only prime numbers less than 100,000

He further considers, without any positive results, common factors of the form $a = 2^n\left(2^{n+1} + 2^k - 1\right)$, for which the second factor is also a prime number. Euler calls this second prime factor f, and with a calculation almost exactly like the one above, concludes that for an amicable pair to be generated, there must be exponents m and n, for which $x = 2^n + 2^{n+1-m}$ and $y = \left(2^{n+1} + 2^{m+n} - 1\right)\left(2^n + 2^{n+1-m}\right)$ for which $m < n + 1$ and all four of the following numbers must be prime:

$$f = 2^{n+1} + 2^{m+n} - 1$$
$$p = x - 1$$
$$q = y - 1$$
$$r = xy - 1$$

This is as far as Euler can go here with analysis, so it is time to examine cases. Taking $m = 1$ yields no amicable pairs.

However, if $m = 2$, it makes

$$f = 3 \cdot 2^{n+1} - 1,\ x = 3 \cdot 2^{n-1} \text{ and } y = 3 \cdot 2^{n-1}\left(3 \cdot 2^{n+1} - 1\right) \text{ and } a = 2^n \cdot f$$

whence

$$p = 3 \cdot 2^{n-1} - 1,\ q = 3 \cdot 2^{n-1}\left(3 \cdot 2^{n+1} - 1\right) - 1 \text{ and } r = 9 \cdot 2^{2n-2}\left(3 \cdot 2^{n+1} - 1\right) - 1.$$

One need only substitute various values of n, hoping to make all four of the numbers f, p, q and r prime. Euler does this in a table:

n =	1	2	3	4	5
f =	11	23	47	95*	191
p =	2	5	11	…	47
q =	32*	137	563	…	9167*
r =	98*	827	6767*	…	…

In this table, numbers that are not prime are marked with a *, and the ellipses mark numbers that were unnecessary to calculate because there is already a composite number in that column. The 98 in column 1 was unnecessary, but easy, so Euler did it anyway.

Only column 2 is free of composite numbers, and this means that it leads to a new amicable pair:

$$\begin{cases} 4 \cdot 23 \cdot 5 \cdot 137 \\ 4 \cdot 23 \cdot 827 \end{cases}$$

This is the first new amicable pair that Euler has shown us how to find, and it was the first new one on his list back in E100.

After this, the fun is over, even though the paper is less than half finished. Euler continues for another 50 pages, doing more forms, more cases, and turning up more and more amicable pairs. At the end of the paper, he summarizes his results, giving 61 amicable pairs, with a couple of typographical errors and a couple of mistakes, and doubling again the world's population of known amicable numbers.

Rather than slog through this, we'll leave it to the interested reader.

Euler wrote a third article with this same title, *De numeris amicabilibus* [E798]. He didn't finish it and it was not published during his lifetime, but was found among his papers and published in 1849, more than 60 years after his death. It is more pedagogical than the other two papers, and he does not give any new amicable pairs.

Amicable numbers are a curious topic. Euler's methods succeeded in reducing a nearly impossible search for special pairs of numbers to a more manageable search. He couldn't guarantee that numbers of a particular form would be amicable, but he made the search small enough that he was able to find quite a few of them. Now, using Euler's methods, but using computers to do the gigantic calculations, thousands of amicable pairs are known.

References:

[E] Euclid, *The Thirteen Books of Euclid's Elements*, 2ed, Sir Thomas Heath, tr., Cambridge University Press, 1926. Also available as a Dover reprint.

[H+W] Hardy, G. H. and E. M. Wright, *An Introduction to the Theory of Numbers*, 3ed, Oxford University Press, 1954.

[E100] Euler, Leonhard, De numeris amicabilibus, *Nova acta eruditorum*, 1747, p. 267-269, reprinted in the *Opera Omnia*, Series I volume 2, p. 59-61. Also available on line at www.EulerArchive.org.

[E134] Euler, Leonhard, Theoremata circa divisores numerorum, *Novi commentarii academiae scientiarum Petropolitanae*, **1** (1747/48), 1750, p. 20-48, reprinted in the *Opera Omnia*, Series I volume 2, p. 62-85. Also available on line at www.EulerArchive.org.

[E152] Euler, Leonhard, De numeris amicabilibus, *Opuscula varii argumenti* **2**, 1750, p. 23-107, reprinted in the *Opera Omnia*, Series I volume 2, p. 86-162. Also available on line at www.EulerArchive.org.

[E798] Euler, Leonhard, De numeris amicabilibus, *Commentationes arithmeticae* **2**, 1849, p. 627-636, reprinted in the *Opera Omnia*, Series I volume 4, p. 353-365. Also available on line at www.EulerArchive.org.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 33 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.