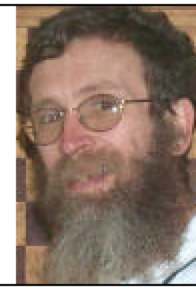
How Euler Did It

by Ed Sandifer



Philip Naudé's problem

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In the days before email, mathematics journals, MathFest, MAA Online and annual Joint Mathematics Meetings, mathematicians had far fewer ways to communicate. Finished works would appear in books and in a few general scientific and scholarly journals like *Acta Eruditorum* or the *Mémoires de l'Academie des Sciences de Berlin*. Euler usually published in the journal of the St. Petersburg Academy, *Commentarii academiae scientiarum Petropolitanae*, or one of its successor journals.

There are other hidden and informal means of scientific communications that are essential to the health of mathematics. These are the ways we answer questions like "What are you working on?" or "What have you discovered that you haven't published yet?" or "What would be interesting or useful to work on next?" Within a single department, we get to ask, and sometimes answer these questions personally. To ask them in a larger community, we have eMail and meetings.

They couldn't do that in Euler's day. Occasionally, someone would go on a "Grand Tour" and visit scientists in several places. We are only sure that Euler did this once, when he stopped to visit Christian Wolff in the 1720's while moving from Basel to St. Petersburg, though he may have made some shorter trips within Germany in the 1740's and 1750's. Many more people traveled to see Euler, but it wasn't enough to sustain a scientific conversation.

Instead, they wrote letters. Over a thousand letters written by Euler survive, and, because he was a very well-organized person, over two thousand letters addressed to Euler still exist. This is why so many of Euler's ideas can be traced to his correspondence. His interest in number theory, for example, began in letters from Goldbach in the 1720's. Other ideas first appear in letters to various Bernoullis, to Stirling, or to Lagrange. Only about a thousand of these letters have been published, though a complete index is in the *Opera Omnia*, Series IV-A, volume 1. Every once in a while, previously unknown letters turn up and the list grows a little bit.

This month's column has its origins in one of those letters. On September 4, 1740, Philip Naudé the younger (1684-1747) wrote Euler from Berlin to ask "how many ways can the number 50 be written as a sum of seven different positive integers?" The problem seems to have captured Euler's imagination. Euler gave his first answer on April 6, 1741, in a paper he read at the weekly meeting of the St. Petersburg Academy. That paper was published ten years later and is number 158 on Eneström's index. Euler solved the problem in a different way in the *Introductio in analysin infinitorum*, E101, published in 1748, and made more improvements in a paper *De partitione numerorum*, "On the partition of numbers," E191, written in 1750, published in 1753. Late in his life, in 1769, he returned to the problem in *De partione numerorum in partes tam numero quam specie datas*, "On the partition of numbers into a given kind or number of parts," E394, published in 1770. Readers at the time might have been a bit confused by the order of the publications. The first one written was the second one published.

All three of the articles, E158, E191 and E394, are reprinted in the *Opera Omnia* volumes on Number Theory, though we will see that they could just as well been classified as papers on Series and put in volumes 14 or 15, or classified as papers on combinatorics and published in volume 7 "Pertaining to the theory of combinations and probability."

The solutions given in E158 and E191 are both based on the relations comparing a series, $a+b+c+$etc., the sum of its powers, $a^n + b^n + c^n + \text{etc.}$, and the sum of products of its terms taken without repeated factors, first two at a time, $ab + ac + bc + ad + bd + cd +$ etc., then three at a time, $abc + abd + acd + bcd + abe +$ etc.

The solutions in E394 and in the *Introductio*, E101, are based on more familiar ideas of multiplying polynomials. We will describe the version given in Chapter XVI of the *Introductio*, "On the Partition of Numbers."

To prepare ourselves for Euler's ideas, let's do a simple example. Let's ask, what is the coefficient of x^5 if we expand $\left(1+x+x^3\right)\left(1+x^2+x^3+x^4\right)\left(x+x^2\right)$? We could answer the question by brute force by finding the ninth degree polynomial that results. On the other hand, since we are only asked to find one of the ten coefficients, we can use the addition properties of exponents, and ask how the exponent 5 can arise. We get a product of degree 5 by adding an exponent from the first factor, 0, 1 or 3, to an exponent from the second factor, 0, 2, 3 or 4, and then adding an exponent from the third factor, 1 or 2, and having the resulting sum be 5. The coefficient of x^5 will be the number of ways of forming this sum. Since there are five ways to do this, (0, 3, 2), (0, 4, 1), (1, 2, 2), (1, 3, 1) and (3, 0, 2). the coefficient of x^5 must be 5.

Conversely, since the product expands to $x+2x^2+2x^3+4x^4+5x^5+4x^6+3x^7+2x^8+x^9$, and since the coefficient of x^5 is 5, there are 5 ways to form the desired sum. This is Euler's basic idea, to use the coefficients of polynomials to count things. It is a remarkably powerful idea, as we shall see.

Euler begins by comparing a product of polynomials in two variables, possibly as an infinite product, with its expansion into a series. He supposes that

$$\left(1+x^a z\right)\left(1+x^b z\right)\left(1+x^g z\right)\left(1+x^d z\right)\ldots = 1+Pz+Qz^2+Rz^3+Sz^4+\ldots$$

and asks for the values of P, Q, R, etc., in terms of a, b,g, etc.

He answers his own question and explains that P is the sum of powers $x^a + x^b + x^g + x^d + \text{etc.}$. Then Q is the sum of the products of powers taken two at a time, R those taken three at a time, etc.

There may be coefficients, though. In P, the coefficient on x^n tells us how many times the number n occurs in the sequence a, b,g, etc. In Q, the coefficient tells us how many different ways n can be formed as the sum of a pair of numbers from the sequence. In R, it tells us how many ways it is the sum of three numbers from the sequence, and so forth.

Got it? Maybe not. Let's do an example. Euler writes, in Chapter 16 of Book 1 of his *Introductio* [E101], "In order that this may become clearer, let us consider the following infinite product."

$$(1+xz)\left(1+x^2z\right)\left(1+x^3z\right)\left(1+x^4z\right)\left(1+x^5z\right)\ldots$$

Euler bravely expands this to get

$$1+z\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+\ldots\right)$$
$$+z^2\left(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+\ldots\right)$$
$$+z^3\left(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+\ldots\right)$$
$$+z^4\left(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+\ldots\right)$$
$$+z^5(\ldots)+\ldots$$

Euler carries the expansion up through z^8 and gives about ten terms of x in each coefficient of z^n. It is quite impressive.

This series is a wealth of information. For example, there is a term $6z^4x^{15}$. The coefficient 6 tells that there are 6 ways to write the exponent of x, 15, as the sum of 4, the exponent of z, different numbers from the sequence 1, 2, 3, 4, 5, … . And, indeed, the four ways are (1, 2, 3, 9), (1, 2, 4, 8), (1, 2, 5, 7), (1, 3, 4, 7), (1, 3, 5, 6) and (2, 3, 4, 6). Similarly, there is a term $15z^7x^{35}$ (Euler includes it, but we didn't). The coefficient 15 tells us that there are 15 ways to write the number 35 as the sum of 7 different positive integers.

Recall that Philip Naudé had asked to write 50 as the sum of 7 different positive integers, so we need only find the coefficient of z^7x^{50}. He doesn't mention this in the *Introductio*, though he tells us un E158 that the required coefficient is 522, and that this is "a most perfect solution to the problem of Naudé."

There is even more information in the series. If we take $z = 1$ and combine like terms, then the coefficient of x^n will be the number of ways to write n as the sum of 1 distinct positive integer, plus the number of ways to write it as the sum of 2 of them, plus the number of ways to write it as the sum of 3 of them, etc. To say this more directly, the coefficient of x^n is the number of ways to write n as the sum of distinct positive integers.

Of course, Euler does an example. If $z = 1$, then the infinite product is

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\text{ etc.}$$

This expands to give

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\text{etc.}$$

Euler points out that the term $6x^8$ tells us that there are 6 ways to write 8 as the sum of distinct positive integers, and he lists them:

8 = 8	8 = 7 + 1	8 = 6 + 2
8 = 5 + 3	8 = 5 + 2 + 1	8 = 4 + 3 + 1

He takes pains to note that we must count the "sum" 8 = 8.

Euler is nowhere near exhausting the idea of using coefficients to count things. He asks, what if we relax the requirement that our summands be distinct, and allow sums like 8 = 3 + 3 + 2? He tells us that "The condition that the numbers be different is eliminated if the product is put into the denominator." This is a bit cryptic. To clarify it a bit, he asks us to consider the expression

$$\frac{1}{(1-x^a z)(1-x^b z)(1-x^g z)(1-x^d z)\ldots}$$

The modern mathematician would ask whether or not this expression converges, but as usual, Euler trusts his notation, and doesn't worry about such things.

The quotient can be expanded into a product of geometric series, which can then be expanded again into a series similar to the one he examined earlier. Euler gives copious details and a good deal of generality, but the case that is most interesting is when a, b, g, d are 1, 2, 3, 4, etc. (The second most interesting case is when they are 1, 3, 5, 7, etc.) Then the quotient

$\frac{1}{1-x^a z} = 1 + xz + x^2z^2 + x^3z^3 + x^4z^4 + \text{etc.}$ The next quotient, $\frac{1}{1-x^b z} = 1 + x^2z + x^4z^2 + x^6z^3 + x^8z^4 + \text{etc.}$ Similarly for the infinitely many other quotients. Again, Euler bravely expands the expression

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)(1-zx^4)\,\text{etc.}}$$

into a series

$$\begin{gathered}1 + z\left(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \ldots\right)\\ +z^2\left(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + \ldots\right)\\ +z^3\left(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + \ldots\right)\\ +z^4\left(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10}\ldots\right)\\ +z^5(\ldots) + \text{etc.}\end{gathered}$$

The coefficients here tell us how many different ways the exponent of x can be written as the sum of the number in the exponent of z positive integers. Euler gives the example, 13 can be written as the sum of 5 whole numbers (not necessarily *different* whole numbers) in 18 different ways because the coefficient of $x^{13}z^5$ turns out to be 18. Euler does not list the 18 different ways.

Now if we take $z = 1$, as before, we get

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\,\text{etc.}} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \text{etc.}$$

The coefficients of this series are called the Partition Numbers, and give the number of ways the corresponding exponent can be written as the sum of any number of positive integers. For example, the term $11x^6$ tells us that 6 can be written as a sum in 11 different ways. Euler lists them for us.

From here, paths lead in many directions. We could look at Euler's work on the partition numbers themselves. He discovered an amazing recursive relation involving the pentagonal numbers. This line of study continued through Ramanujan, who discovered a number of beautiful and astonishing properties.

Instead of dwelling on this, we will only sketch a few other results and mention a bit about where they lead.

One thing Euler does, but we won't give details, is to show that the coefficients of the expansion

$$\frac{x^3}{(1-x)(1-x^2)} = x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + \text{etc.}$$

give the number of ways that a number can be written as the sum of two different numbers. For example, $4x^9$ tells us that there are 4 ways to write 9 as the sum of two different numbers. They are 1+8, 2+7, 3+6 and 4+5. Likewise, $3x^8$ corresponds to the sums 1+7, 2+6 and 3+5.

Then he shows that the coefficient of x^n in the expansion also gives the number of ways that n-3 can be given as the sum of 1's and 2's. Here, $3x^8$ tells us that there will be 3 ways to write 8-3=5 as a sum of 1's and 2's. They are 1+1+1+1+1, 1+1+1+2 and 1+2+2. Note that it is not necessary to include both 1's and 2's, so 1+1+1+1+1 is an admissible sum.

Euler also showed how to find similar relations between the number of sums of k different numbers and the number of sums involving only the numbers 1 through k. He did all of his analysis algebraically.

We could also explore the other ways that Euler used the tools of generating functions that he first developed in his study of partitions in E158.

In 1878, J. J. Sylvester, then teaching at Johns Hopkins, and his student Fabian Franklin, working with a group of other graduate students at Hopkins, gave a wonderful graphical proof of the same result, as well as a number of new results. Their paper was based on the insight that arrangements of 22 stars like the following,

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can be viewed as giving the number 22 as a sum of three numbers, 6 + 6 + 4, or as a sum involving only numbers 1 through 3, here 3 + 3 + 3 + 3 + 2 + 2.

Rather than follow these paths now, we will leave them for another column.

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How Euler Did It is updated each month.