

Finding logarithms by hand

July 2005

Today, it is just as easy to take a square root as it is to find a logarithm. You just find your calculator, turn it on, and press a few buttons. The first step of that process usually takes the longest. In 1748, though, when Euler published the *Introductio in analysin infinitorum*, most mathematicians and scientists were quite good at taking square roots by hand, but logarithms required difficult analysis or a book of tables. The seven-place tables of Briggs and Vlacq had been available for over a hundred years, but the ten place tables of Jurij Vega were still fifty years in the future.

In the *Introductio*, perhaps Euler's most interesting and influential work, Euler did not append tables of anything, be it logarithms, square roots, or trigonometric functions. So, when he covered this material (except for square roots, which he assumed we already know) he wanted to make sure his readers knew how to find the values of the functions, even if in practice they would be looking them up in tables. For the trigonometric tables, Euler gave fast converging series and a few tricks involving trig identities that gave values to 20 or more decimal places in just a dozen or so steps. In Chapter VI, "On Exponentials and Logarithms," though, he gives an algorithm that does not use series.

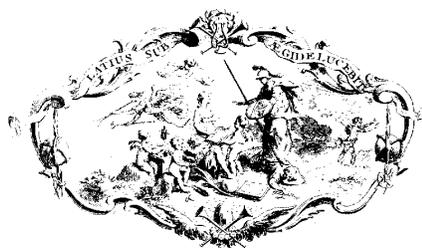
Euler's algorithm is based on the elementary properties of logarithms and the fact that $\log x$ is a monotone increasing function of x .

Notation in 1748 was a bit awkward by modern standards. Euler always wrote $l x$ where we would write $\log x$ or $\ln x$, and he always has to tell us what the base of the logarithm is. We will use the modern notation.

Euler begins his description of the algorithm by telling us "putting $\log y = z$ & $\log v = x$, then

INTRODUCTIO
IN ANALYSIN
INFINITORUM.
 AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & Academie Imperialis Scientiarum PETROPOLITANÆ Socio.

T O M U S P R I M U S.



LAUSANNÆ,
 Apud MARCUM-MICHAELEM BOUSQUET & Socios.

 M D C C X L V I I I

$\log\sqrt{vy} = \frac{x+z}{2}$. That is, geometric means of the arguments correspond to arithmetic means of the logarithms.

Euler prepares us for his algorithm by explaining, “if a given number b lies between limits a^2 and a^3 , the logarithms of which are 2 and 3 respectively, then we will find the value of $a^{2\frac{1}{2}}$, that is $a^2\sqrt{a}$, and then b must lie either between the limits a^2 and $a^{2\frac{1}{2}}$, or between $a^{2\frac{1}{2}}$ and a^3 .” Euler is setting us up for what we now call a binary search.

For his example, Euler takes $a = 10$, so we will be finding base ten logarithms. He takes $b = 5$, and plans to find $\log 5$. He notes that 5 is between 1 and 10, so $\log 5$ is between 0 and 1.

Now, the geometric mean of 1 and 10 has as its logarithm the arithmetic mean of 0 and 1. With this, Euler begins a binary search. He writes

$$\begin{array}{lll} A = 1.000000 & \log A = 0.000000 & \\ B = 10.000000 & \log B = 1.000000 & C = \sqrt{AB} \\ C = 3.162277 & & \end{array}$$

Now, using the $\log\sqrt{vy} = \frac{x+z}{2}$ fact, since C is the geometric mean of A and B , its logarithm is the arithmetic mean of $\log A$ and $\log B$, that is 0.500000. He adds this to his table and gets

$$\begin{array}{lll} A = 1.000000 & \log A = 0.000000 & \\ B = 10.000000 & \log B = 1.000000 & C = \sqrt{AB} \\ C = 3.162277 & \log C = 0.500000 & \end{array}$$

Since 5 is between $B = 10.000000$ and $C = 3.162277$ (and not between A and C), $\log 5$ is between $\log B$ and $\log C$. He takes repeats his first step and takes D to be the geometric mean of B and C , so its log is the arithmetic mean. Euler’s table grows to be

$$\begin{array}{lll} A = 1.000000 & \log A = 0.000000 & \\ B = 10.000000 & \log B = 1.000000 & C = \sqrt{AB} \\ C = 3.162277 & \log C = 0.500000 & D = \sqrt{BC} \\ D = 5.623413 & \log D = 0.750000 & \end{array}$$

Euler continues this process of bisection and geometric means until he runs out of letters. Eventually, he gets $X = 4.999997$, $Y = 5.000003$ and $Z = 5.000000$, and $\log z = 0.6989700$. Euler’s whole table is shown below:

LIB. I.	$A = 1, 000000;$	$lA = 0, 0000000$	fit
—	$B = 10, 000000;$	$lB = 1, 0000000;$	$C = \sqrt{AB}$
	$C = 3, 162277;$	$lC = 0, 5000000;$	$D = \sqrt{BC}$
	$D = 5, 623413;$	$lD = 0, 7500000;$	$E = \sqrt{CD}$
	$E = 4, 216964;$	$lE = 0, 6250000;$	$F = \sqrt{DE}$
	$F = 4, 869674;$	$lF = 0, 6875000;$	$G = \sqrt{DF}$
	$G = 5, 232991;$	$lG = 0, 7187500;$	$H = \sqrt{FG}$
	$H = 5, 048065;$	$lH = 0, 7031250;$	$I = \sqrt{FH}$
	$I = 4, 958069;$	$lI = 0, 6953125;$	$K = \sqrt{HI}$
	$K = 5, 002865;$	$lK = 0, 6992187;$	$L = \sqrt{IK}$
	$L = 4, 980416;$	$lL = 0, 6972656;$	$M = \sqrt{KL}$
	$M = 4, 991627;$	$lM = 0, 6982421;$	$N = \sqrt{KM}$
	$N = 4, 997242;$	$lN = 0, 6987304;$	$O = \sqrt{KN}$
	$O = 5, 000052;$	$lO = 0, 6989745;$	$P = \sqrt{NO}$
	$P = 4, 998647;$	$lP = 0, 6988525;$	$Q = \sqrt{OP}$
	$Q = 4, 999350;$	$lQ = 0, 6989135;$	$R = \sqrt{OQ}$
	$R = 4, 999701;$	$lR = 0, 6989440;$	$S = \sqrt{OR}$
	$S = 4, 999876;$	$lS = 0, 6989592;$	$T = \sqrt{OS}$
	$T = 4, 999963;$	$lT = 0, 6989668;$	$V = \sqrt{OT}$
	$V = 5, 000008;$	$lV = 0, 6989707;$	$W = \sqrt{TV}$
	$W = 4, 999984;$	$lW = 0, 6989687;$	$X = \sqrt{WV}$
	$X = 4, 999997;$	$lX = 0, 6989697;$	$Y = \sqrt{VX}$
	$Y = 5, 000003;$	$lY = 0, 6989702;$	$Z = \sqrt{XY}$
	$Z = 5, 000000;$	$lZ = 0, 6989700;$	

Note that the logarithms are given to seven places, just as in the tables by Briggs and Vlacq.

Euler then shows how $\log 2$ is easily found as $1 - \log 5$ and notes that with these two values it is now easy to find the logs of 4, 8, 16, 32, 64, etc., as well as 20, 40, 80, 25, 50, etc.

This lesson on finding logarithms by hand ends with this. In the next chapter, Chapter VII titled “Exponentials and Logarithms Expressed through Series,” Euler does use the series methods we see more often today. These are similar to the series methods that Briggs had used over a hundred years earlier. We will briefly describe Euler’s methods.

Euler has switched to the natural logarithm, which he still denotes $l x$, and mentions in the text that the base is e . First, Euler finds the well-known series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

Then, $\log(1-x)$ has almost the same series, but it does not alternate. Using the elementary properties of logarithms, we get to subtract the two series. This eliminates all the terms of even degree and gives

$$\ln\left(\frac{1+x}{1-x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \text{etc.}$$

For small values of x , this last series converges rapidly. If we pick clever values of x , we can start to construct tables of logarithms. For example, if $x = 1/5$, we get $\ln \frac{6}{4}$. Likewise, taking $x = 1/7$ and $x = 1/9$ gives the logs of $4/3$ and $5/4$, respectively. Finally, we combine these to get $\ln 2 = \ln \frac{3}{2} + \ln \frac{4}{3}$, $\ln 3 = \ln \frac{3}{2} + \ln 2$ and $\ln 4 = 2 \ln 2$. Similar tricks give logarithms for 5, 6, 8, 9 and 10, all from just three series.

Note that we missed the logarithm of 7. This requires a famous trick that Briggs used for base 10 logarithms. Now Euler repeats it for natural logarithms. Take $x = 1/99$ and get $\ln \frac{100}{98} = \ln \frac{50}{49}$. Since we already have enough information to find $\ln 50$, it is now easy to find $\ln 49$, and half of that is $\ln 7$. Very clever. Euler shows off his method by finding logarithms of the integers from 1 to 10 to twenty-five decimal places.

At the end of Chapter VI there are six examples, five of which are “word problems.” They are the only word problems in the entire *Introductio*, and there are no word problems at all in the two sequels, *Calculus differentialis* and *Calculus integralis*, so these are worth special attention.

Example 1 is not a word problem, but it has a word problem behind it. The problem is “to find the value of the power $2^{\frac{7}{12}}$.” The solution is very simple by Eulerian standards, and the answer turns out to be 1.498307.

Euler doesn’t mention it, but it is clear why he chose this problem. It is because the answer is so close to 1.5. This is important in the theory of music. If we vibrate two strings of equal density and tension, where one string is 1.5 times as long as the other, then the notes they sound will differ by what musicians call a “fifth.” The scales built on this “perfect fifth” are called “Pythagorean,” for reasons that are interesting, but too lengthy for this column. The notes on a Pythagorean scale are not quite evenly spaced. Euler, who studied the physics of music with great enthusiasm, [G] proposed an “even tempered” scale, for which the notes are spaced by making the ratio of the lengths of two consecutive strings be $\sqrt[12]{2}$. This example shows that the fifth note up on this scale differs from a Pythagorean fifth by only about one tenth of one percent.

Euler’s second problem reads “If the number of people who inhabit a certain province grows by one part in 30 each year, and initially the province has 100,000 people, find the number of inhabitants after 100 years. Euler uses the elementary properties of logarithms to find that the logarithm of the answer is $100 \log \frac{31}{30} + \log 100,000 = 6.4240439$, so the number of inhabitants will be 2,654,874.

Today this is a routine problem in exponential growth, but in 1748 it was one of the first hints that mathematics might be useful in the social sciences. At the time, Berlin was one of the world’s few large cities to do a census. A careful census in 1747 showed the population of Berlin to be 107,224. [L] Other cities tried to estimate their populations using birth and death records, and such methods suggested that London had grown from about 100,000 to about a million over the previous 100 years. This problem, and the three that follow, give mathematical demonstrations that such ten-fold growth in a century is possible, even for large populations like that of Berlin.

Euler's next three examples are also population questions:

“After the Flood, all people are descended from six people. If we suppose that after 200 years the population had grown to 1,000,000, then by what part must the population grow each year?”

Euler is asking for a growth rate. Today we would probably express the growth rate as a percent.

Euler sets up the problem to find x if $6\left(\frac{1+x}{x}\right)^{200} = 1000000$. He finds a rate of about 1/16 per year, that we would write as 6.25% per year.

The problem is to find the annual growth rate if the population doubles in a century. Here again, Euler expresses his growth rate as a fraction, 1/144, rather than the percentage, about 0.7%, as we probably would today.

The last population problem is to find how long it will take the population to double if the growth rate is 1/100 per year. Euler takes logarithms to get 231 years.

Euler's last word problem is a compound interest problem, and is kind of tricky. Somebody borrows 400,000 florins at an “annual rate of usury of 5 percent.” (They didn't call it “interest” yet.) If he pays 25,000 florins each year, then how long will it take to repay the debt? Euler solves this by letting a be the 400,000 florins in the debt and b the 25,000 florins in each payment. Then he writes the

debt after one year as $\frac{105}{100}a - b$, after two years as $\left(\frac{105}{100}\right)^2 a - \left(\frac{105}{100}\right)b - b$. Now he writes $n = \frac{105}{100}$ and x as the number of years, and gets the debt after x years as

$$n^x a - n^{x-1}b - n^{x-2}b - n^{x-3}b - \dots - b.$$

Euler sees the geometric series in this and writes it as

$$n^x a - b(1 + n + n^2 + \dots + n^{x-1})$$

The sum of the finite geometric series inside the parentheses is $\frac{n^x - 1}{n - 1}$. Since we want to find the

value of x that makes the debt equal to zero, we get the equation $n^x a = \frac{n^x b - b}{n - 1}$. After substituting the given values for n , a and b , we use logarithms to find that x is a little less than 33, so the debt is paid off after 33 years.

Today, this seems like a rather difficult but otherwise fairly routine problem in the mathematics of finance.

Textbooks have changed a good deal in the last 250 years. Euler's *Introductio* was a new kind of textbook in its day. Its presentation of logarithms is very similar to the one we use today, though the actual calculation of logarithms became obsolete with the advent of calculators. It is comforting to know, though, that Euler did not invent the “fake” word problem. His very few word problems were real problems that addressed actual issues that were important and interesting in the 1740s. It would be nice if modern authors could craft their products so carefully.

References:

- [E101] Euler, Leonhard, *Introductio in analysin infinitorum*, 2 vols., Bosquet, Lucerne, 1748, reprinted in the *Opera Omnia*, Series I volumes 8 and 9. English translation by John Blanton, Springer-Verlag, 1988 and 1990. Facsimile edition by Anastaltique, Brussels, 1967.
- [G] Greated, Clive, "Leonhard Euler," *The New Grove Dictionary of Music and Musicians*, 2ed., Macmillan, New York, 2001 volume 8, pages 415-416
- [L] Lewin, C. G., *Pensions and Insurance before 1800: A Social History*, Tuckwell Press, East Lothian, Scotland, 2003.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 33 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.
Copyright ©2005 Ed Sandifer