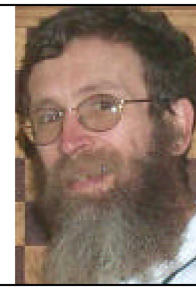
How Euler Did It

by Ed Sandifer



Wallis's formula

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Besides everything else he did, Euler was the best mathematics textbook writer of his age, with a line of texts that extended from arithmetic to advanced calculus, and a popular book on general science as well. We summarize his textbook output below:

1738	*Rechenkunst*, Arithmetic, in German and Russian, 277 pages
1744	*Methodus inveniendi*, Calculus of variations, Latin, 322 pages
1748	*Introductio in analysin infinitorum*, Precalculus, Latin, 2 vol, 320 + 398 pages
1755	*Calculus differentialis*, Differential calculus, Latin, 880 pages
1768	*Lettres à un Princesse d'Allemagne*, Letters to a German princess, general science, French, 3 vol, 314 + 340 + 404 pages
1768	*Calculus intgralis*, Integral calculus, Latin, 3 vol, 542 + 526 + 639 pages
1770	*Vollständige Anleitung zur Algebra*, German, 2 vol, 356 + 532 pages

That is quite a curriculum, and, at over 5800 pages and thirteen volumes quite a bookshelf. The calculus alone, differential and integral, is over 2500 pages, about twice the length of the larger modern texts, and Euler didn't include exercises. To be fair, it includes a lot of differential equations, but much of the material on series was covered in the precalculus text, the *Introductio*.

This month, celebrating the beginning of the second year of this column, we will look at a couple of paragraphs about infinite products from the middle of the first volume of the *Calculus integralis*, in a section titled "De evolutione integralium per producta infinita," or "On the expansion of integrals by infinite products."

The three volumes of the *Calculus Integralis* are divided into parts, which are, in turn, divided into sections, then chapters and finally paragraphs. The first two volumes number 1275 paragraphs and 173 problems. The paragraph numbers and problem numbers start again at 1 for the third volume.

The *Calculus integralis* is, for the most part, organized as a series of problems and their solutions and generalizations. A "problem" is likely to have three or four corollaries. The section "De evolutione integralium per producta infinita" begins with Problem 43 and paragraph 356 (both counting from the beginning of the book):

356. To expand the value of this integral $\int \frac{dx}{\sqrt{1-xx}}$ in the case $x = 1$.

To a modern reader, this isn't very clear. It looks like an indefinite integral, but this is how Euler wrote definite integrals. He means us to take the particular antiderivative that is zero at the left hand endpoint, in this case assumed to be zero, and then to evaluate this antiderivative at $x = 1$. So, we would write Euler's integral as $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. It is easy to see that this is an arcsine and it has value $\frac{p}{2}$. That's not the point, though.

For the preceding several sections, Euler has been doing gymnastics with integration by parts, so it fits in very naturally to use integration by parts here, not on the specific $\int \frac{dx}{\sqrt{1-xx}}$, but on the more general $\int \frac{x^{m-1}dx}{\sqrt{1-xx}}$. He gets $\int \frac{x^{m-1}dx}{\sqrt{1-xx}} = \frac{m+1}{m}\int \frac{x^{m+1}dx}{\sqrt{1-xx}}$. We are still doing definite integrals, and the leading term outside the integral evaluates to zero at both endpoints, so that term disappears.

Applying this repeatedly, starting with the case $m = 1$, so $m - 1 = 0$ gives

$$\begin{aligned}\int \frac{dx}{\sqrt{1-xx}} &= \frac{2}{1}\int \frac{xxdx}{\sqrt{1-xx}}\\ &= \frac{2\cdot 4}{1\cdot 3}\int \frac{x^4\,dx}{\sqrt{1-xx}}\\ &= \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\int \frac{x^6\,dx}{\sqrt{1-xx}}\end{aligned}$$

Now, Euler makes a rather suspicious step, taking i to be an infinite number, extends this to infinity, and writes the infinite product

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot 2i}{1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot (2i-1)}\int \frac{x^{2i}dx}{\sqrt{1-xx}}$$

Now, unexpectedly, we do the same thing with $\int \frac{xdx}{\sqrt{1-xx}}$, another definite integral which is known to be equal to 1. We get another equally dubious infinite product

$$\int \frac{xdx}{\sqrt{1-xx}} = \frac{3\cdot 5\cdot 7\cdot 9\cdot \ldots \cdot (2i+1)}{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot 2i}\int \frac{x^{2i+1}\,dx}{\sqrt{1-xx}}$$

These products are a bit awkward, since the fraction part diverges to infinity, while the integral part goes to zero. Nineteenth century analysis taught us that such objects ought to be treated carefully. However, one of the principle motivations for that analysis was to understand why sometimes, as in the present case, manipulations that used infinite numbers would work, and lead to correct and consistent results, but that sometimes they would lead to contradictions. However, those are not the issues of 1768 when Euler published the *Calculus integralis*.

Euler asks us to observe that, if i is an infinite number, then these last factors of the two infinite products, $\int \frac{x^{2i}dx}{\sqrt{1-xx}}$ and $\int \frac{x^{2i+1}dx}{\sqrt{1-xx}}$ will be equal. This allows Euler to know the ratios between the two infinite products. He leads us to the ratio as follows:

Let $\frac{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot 2i}{1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot (2i-1)} = M$ and $\frac{3\cdot 5\cdot 7\cdot 9\cdot \ldots \cdot (2i+1)}{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot 2i} = N$. Then, the old ":" notation for ratios,

$$\int \frac{dx}{\sqrt{1-xx}} : \int \frac{xdx}{\sqrt{1-xx}} = M : N = \frac{M}{N} : 1$$

We know that the ratio of the two integrals is $\frac{p}{2}$, so the ratio of M to N ought to be the same thing. Writing that ratio as an infinite product, we get

$$\frac{p}{2} = \frac{M}{N} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{etc.}$$

This is the well-known Wallis Formula, first discovered by John Wallis (1616-1703) [O'C]. Wallis was a part-time cryptographer for both sides of the religious wars in England, and Savilian Professor of Geometry at Oxford. He published this result in his most famous work, the book *Arithmetica infinitorum*, published in 1656, ten years before Newton discovered calculus, and almost 30 years before Leibniz published his results. In its time, Wallis's infinite product was so remarkable that some important mathematicians, including the great Christian Huygens, simply didn't believe it.

Euler had used Wallis's formula before, in 1729, when he discovered what we now call the Gamma function. There he reduced the Gamma function to an infinite product and compared that infinite product to Wallis's formula to find that $\Gamma\left(\frac{1}{2}\right) = \sqrt{p}$. Now, almost 40 years later and over a hundred years after Wallis, Euler uses Wallis's formula again to check that his bold calculations with ratios of infinite quantities are working.

References:

[O'C] O'Connor, J. J., and E. F. Robertson, "John Wallis", The MacTutor History of Mathematics archive, http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Wallis.html, February, 2002.

[E242] Euler, Leonhard, *Institutiones calculi integralis*, Vol. 1, St. Petersburg, 1768, reprinted in *Opera Omnia*, Series I volume 11, Birkhäuser, 1913.

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How Euler Did It is updated each month.