

## How Euler Did It

 by Ed Sandifer

## Arc length of an ellipse

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It is remarkable that the constant, $\pi$, that relates the radius to the circumference of a circle in the familiar formula $C=2 \pi r$ is the same constant that relates the radius the area in the formula $A=\pi r^{2}$. This is a special property of circles. Ellipses, despite their similarity to circles, are quite different. There is no easy relationship between the circumference and the area of an ellipse.

On the one hand, if the two semi-radii of an ellipse are $a$ and $b$, then the area of the ellipse is given by $A=\pi a b$. The constant $\pi$ is the same constant that works for circles. The area of a circle is a special case of this. On the other hand, arc length on an ellipse is a deep and considerably more difficult question. As we will see, the arc length is given either by a hard integral or by a rather formidable series. Early work was done by the Italian mathematician Fagnano and the Swede Klingenstierna, but we will follow Euler's version.

Euler worked throughout his life on integrals involving the arc length of the ellipse. We will look at his earliest efforts, a paper written in 1732, published in 1738, in which he found a series for the arc length of a quarter of an ellipse. The result is part of paper number 28 on the Eneström index, and is titled "Specimen de constructione aequationum differentialium sine indeterminatarum separatione," or, in English, "Example of the construction of a differential equation without the separation of variables." As the title suggests, the arc length of the ellipse arises as Euler is pursuing a problem in differential equations.

As always, we begin with notation. In Fig. 1, arc $B M C$ is a quarter of an ellipse, and other parts are defined as follows:
$A C=a$, the major axis of the ellipse
$B C=b$, the minor axis of the ellipse
$A T$ is the tangent to the ellipse at $A$
$C T$ cuts the ellipse at $M$
$A M=s$ is the length of the arc $A M$
$A T=t$
$C P=x$


Fig. 1.

Euler plans to use differentials on this diagram, so he means the distance $t T$ and the arc $m M$ to be very small. Also, we are to assume that $m p$ and $M P$ are perpendicular to the axis $C A$.

Note that Euler uses the variable $t$ twice here, once as a point and once as a length. This was a common practice in the eighteenth century and it often gets confusing.

We are talking about an ellipse that today we would describe with the equation $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$.
Euler gives an equivalent form, $P M=\frac{b \sqrt{a^{2}-x^{2}}}{a}$, and notes that, by similar triangles, $t x=b \sqrt{a^{2}-x^{2}}$ or, equivalently, $x=\frac{a b}{\sqrt{b b+t t}}$.

With a bit of work, we can take $y=P M$ and know that the arc length differential is given by $d s=\sqrt{1+\left(y^{\prime}\right)^{2}}$ to find that $d s=\frac{-d x \sqrt{a^{4}-\left(a^{2}-b^{2}\right) x^{2}}}{a \sqrt{a^{2}-x^{2}}}$. We could try to integrate this between 0 and $a$ to find the length of the arc $B M A$, but Euler and others have learned from experience that this doesn't work very well. Instead, Euler replaces $x$ with $t$. This gives

$$
d s=\frac{b d t \sqrt{b^{4}+a^{2} t t}}{(b b+t t)^{3 / 2}}
$$

Now the arc length $B M A$ is the integral of this from 0 to $\infty$.
Before he gets down to the integral, Euler wants to make one more substitution. If you think about it, the ratio of the axes of an ellipse, $a / b$ tells us how much the ellipse is like a circle. If the ratio is close to 1 , then the ellipse is more circular. Euler wants, instead, a measure of how different the ellipse is from a circle. He defines a measure $n$ by the equation $a^{2}=(n+1) b^{2}$. Here, when $n$ is close to zero, then $a$ is close to $b$ and the ellipse is not much different from a circle. If we use this to replace $a$ with $n$, we get

$$
d s=\frac{b^{2} d t \sqrt{\left(b^{2}+t^{2}\right)+n t^{2}}}{\left(b^{2}+t^{2}\right)^{3 / 2}}
$$

This doesn't look like progress, but Euler has a surprise, one that Newton had used over 50 years earlier. Remember the binomial theorem:

$$
\begin{aligned}
(x+y)^{m} & =x^{m}+\binom{m}{1} x^{m-1} y+\binom{m}{2} x^{m-2} y^{2}+\binom{m}{3} x^{m-3} y^{3}+\ldots \\
& =x^{m}+\frac{m}{1} x^{m-1} y+\frac{m(m-1)}{1 \cdot 2} x^{m-2} y^{2}+\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^{m-3} y^{3}+\ldots
\end{aligned}
$$

We usually use this for m a positive integer, and, in that case, the numerators in the second formula eventually are zero and we get a finite series. Newton showed that the theorem is still true for fractional values of $m$, but the result is an infinite series. Euler takes $m=1 / 2$ and denotes the coefficients by

$$
\begin{aligned}
& A=\frac{1}{2} \\
& B=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1 \cdot 2}=\frac{-1}{2} \cdot \frac{1}{4} \\
& C=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1 \cdot 2 \cdot 3}=\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}, \mathrm{etc} .
\end{aligned}
$$

Now he applies the binomial theorem to the radical in the numerator of $d s$ to get

$$
\left(b^{2}+t^{2}\right)^{1 / 2}+\frac{A n t^{2}}{\left(b^{2}+t^{2}\right)^{1 / 2}}+\frac{B n^{2} t^{4}}{\left(b^{2}+t^{2}\right)^{3 / 2}}+\frac{C n^{3} t^{6}}{\left(b^{2}+t^{2}\right)^{5 / 2}}+\text { etc. }
$$

so that

$$
d s=\frac{b^{2} d t}{b^{2}+t^{2}}+\frac{A b^{2} n t^{2} d t}{\left(b^{2}+t^{2}\right)^{2}}+\frac{B b^{4} n^{2} t^{4} d t}{\left(b^{2}+t^{2}\right)^{3}}+\frac{C b^{6} n^{3} t^{6} d t}{\left(b^{2}+t^{2}\right)^{4}}+\text { etc. }
$$

So, the length of the $\operatorname{arc} A M B$ will be the integral of this series from 0 to $\infty$. Notice how nicely Euler's trick got rid of the radical in the denominator, and also how introducing the term $n$ helps the series converge rapidly for small values of $n$.

The first term of this series integrates as an arctangent to give $b \frac{\pi}{2}$. The rest of the terms reduce, as if by magic, to the first term using integration by parts. The second term, for example, (ignoring $A$ and $n$ to make it a little easier to type) gives

$$
\int \frac{b^{2} t^{2} d t}{\left(b^{2}+t^{2}\right)^{2}}=\frac{1}{2} \int \frac{b b d t}{b b+t t}-\frac{1}{2} \frac{b^{2} t}{b b+t t}
$$

where in the integration by parts, we took $d v=\frac{2 t d t}{\left(b^{2}+t^{2}\right)^{2}}$ so that $v=\frac{-1}{b^{2}+t^{2}}$.
Similarly, the third term reduces to the second, and the fourth to the third, and we get, after a few pages of calculations,

$$
\int \frac{b^{2} t^{4} d t}{\left(b^{2}+t^{2}\right)^{3}}=\frac{1 \cdot 3}{2 \cdot 4} \int \frac{b^{2} d t}{b b+t t}-\frac{1 \cdot 3}{2 \cdot 4} \frac{b^{2} t}{b b+t t}-\frac{1}{4} \frac{b^{2} t^{3}}{(b b+t t)^{2}}
$$

amd

$$
\int \frac{b^{2} t^{6} d t}{\left(b^{2}+t^{2}\right)^{4}}=\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \int \frac{b^{2} d t}{b b+t t}-\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{b^{2} t}{b b+t t}-\frac{1 \cdot 5}{4 \cdot 6} \frac{b^{2} t^{3}}{(b b+t t)^{2}}-\frac{1}{6} \frac{b^{2} t^{5}}{(b b+t t)^{3}} .
$$

Euler is a genius at such calculations, and he tells us that from this, "the law for the integrals of the remaining terms is apparent enough."

The information Euler needs to find the length of the arc $A M B$, is hidden in these series. He points out that when $t=0$ or $t=\infty$, the "algebraic" terms, that is, the terms outside the integrand, are all zero, so we only have to worry about the integrals themselves. With those swept away, the pattern for the reduction of the integral becomes clear:

$$
\int \frac{b^{2 m} t^{2 m} d t}{\left(b^{2}+t^{2}\right)}=\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot(2 m-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2 m} \frac{\pi b}{2}
$$

We are almost done. Euler takes $e=\frac{\pi b}{2}$. This looks odd to us, but Euler had not yet adopted the convention that the symbol $e$ always denotes the base for the natural logarithms. Substitute these values back in the integral of $d s$, and putting the parameter $n$ and the coefficients $A, B, C$, etc., back into the equation, we get

$$
A M B=e\left(1+\frac{1}{2} A n+\frac{1 \cdot 3}{2 \cdot 4} B n^{2}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \mathrm{Cn}^{3}+\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \mathrm{Dn}^{4}+\text { etc. }\right) .
$$

If we also substitute for $A, B, C, D$, we get

$$
A M B=e\left(1+\frac{1 \cdot n}{2 \cdot 2}-\frac{1 \cdot 3 \cdot n^{2}}{2 \cdot 2 \cdot 4 \cdot 4}+\frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot n^{3}}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}-\frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot n^{4}}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}+\text { etc. }\right)
$$

This is Euler's answer, a rather intimidating series. We might want to replace the symbol $e$ with its value and write it as

$$
A M B=\frac{\pi b}{2}\left(1+\frac{1 \cdot n}{2 \cdot 2}-\frac{1 \cdot 3 \cdot n^{2}}{2 \cdot 2 \cdot 4 \cdot 4}+\frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot n^{3}}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}-\frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot n^{4}}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}+\text { etc. }\right) .
$$

We can check that, when $n=0$ and $a=b=1$, we get the answer we expect, $\pi / 2$, and that as $a$ increases, $n$ also increases, as does the value of the series, so the answer at least makes sense, even if it isn't as simple as we might have hoped.

Over the last 270 years, we have learned a lot more about arc lengths on ellipses. Euler himself added a good deal more to the subject, including the so-called addition formula for elliptic integrals. These arc lengths are the foundation of deep and rich studies of elliptic integrals, elliptic curves and elliptic functions, with applications across a vast spectrum of mathematics, from mechanics to Wiles' solution of Fermat's Last Theorem. It all has roots in this paper, and the curious fact that arc length for ellipses is so much more complicated than it is for circles.

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