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How Euler Did It

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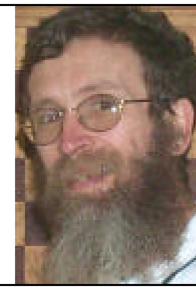

Cramer's Paradox

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Gabriel Cramer (1704-1752) is best known for Cramer's rule, a technique for solving simultaneous linear equations that is not very useful in practice, but is of immense theoretical importance in linear algebra. We will not be talking much about Cramer's rule here, though. Instead, we will discuss a question he asked of Euler, and how Euler answered it. That question got the name "Cramer's Paradox."

Cramer (his portrait is at the right) lived and worked in Geneva. One could do a compare-and-contrast using Euler and Cramer, both Swiss, but one spoke French at home, and the other German. Both earned graduate degrees very early with theses on sound, but one stayed in Switzerland while the other left and never returned. Cramer was born three years earlier, but died 31 years earlier, in 1752 at the age of only 47. Euler and Cramer began corresponding in 1743. At least 16 of their letters survive (one was discovered just last year), the last one just two months before Cramer's death, and Cramer's son Philip continued the correspondence for another year.

Before we dive into Cramer's Paradox, we'd better make sure we have our definitions straight. Cramer and Euler were interested in real algebraic curves, curves defined by equations of the form $f(x,y)=0$, where f is a polynomial of degree n. The special case $n = 3$ comes up later, so it is useful to note that third degree algebraic curves, what we call cubic curves, have the form

$$\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3 + \varepsilon x^2 + \zeta xy + \eta y^2 + \theta x + \iota y + \kappa = 0.$$

There are ten coefficients to the cubic curve, the ten Greek letters given. Since for any non-zero constant c, $cf(x,y)=0$ and $f(x,y)=0$ define the same curve, we can pick any value we like for one of the (non-zero) coefficients, and then the other nine coefficients are determined by the curve.

Several of the letters that Euler and Cramer exchanged dealt with questions about algebraic curves. At the time, mathematicians all believed, but were still unable to prove, what we now call the

Fundamental Theorem of Algebra, that any polynomial of degree *n* has at most *n* real roots, and that, counting complex roots and multiplicities, it will have exactly *n* roots. Euler and d'Alembert both gave proofs that were accepted in the 1750's, but a proof that satisfies modern standards of rigor would have to wait for Gauss, in his doctoral thesis.

In one of those letters, dated September 30, 1744, Cramer asked the question that would soon bear his name, though some people trace its origins at least to Maclaurin, 15 or 20 years earlier. Cramer stated two "facts" about cubic curves:

1. One curve of order 3 (what we call *degree* he calls *order*, and hereafter we will try to use his term) can be determined by 9 given points, and
2. Two curves of order 3 can intersect in 9 points, thus one may find two curves of order 3 passing through these given nine points.

He asked how these can both be true, since the 9 points of intersection seem to determine two different curves of order 3.

Euler worked on the question on and off for the next couple of years, and put his answer into an article that he wrote in 1747, *Sur une contradiction apparente dans la doctrine des lignes courbes* (On an apparent contradiction in the rule of curved lines.) That article was published in 1748, and is number 147 on Eneström's index of Euler's works.

Euler begins his article with a "teaser." Though this is a real work of scholarship that addresses one of the important problems of the day, Euler shows a bit of a mischievous spirit in this and other passages throughout the paper:

> One believes generally that Geometry distinguishes itself from the other sciences because every advance is founded upon the most rigorous proofs, and that one will never find an occasion for controversy.

Of course, Euler is having fun here, since he plans to exploit a point where "Geometry" (what we call rigorous mathematics) was not quite rigorous at the time, and create what seems like a paradox or a controversy. He goes on to write

> I am going to describe two propositions of Geometry, both rigorously demonstrated, which would seem to lead to an open contradiction.

Euler restates Cramer's two "facts" and their generalizations. He reminds us that two points determine a line, a curve of order 1 (even though a line, in standard form $ax + by + c = 0$, has *three* coefficients, a, b and c.) Five points determine a conic (order 2), and nine points determine a cubic. In general, Euler tells us that a curve of order *n* is determined by $\frac{n^2+3n}{2}$ points.

Euler calls this last claim Proposition 1. His proof is based on writing the general equation of a curve of order *n* as

$$Ay^n + (B+Cx)\,y^{n-1} + (D+Ex+Fx^2)\,y^{n-2} + \left(G+Hx+Ix^2+Kx^3\right)y^{n-3} + \text{etc.} = 0$$

This has $\frac{n^2+3n}{2}+1$ coefficients (the (n+1)st triangular number, and so is determined by $\frac{n^2+3n}{2}$ linear equations. Since each given point provides us with an equation, and since the ratios of the coefficients that determine a curve, we need $\frac{n^2+3n}{2}$ points to determine the curve.

If you see a flaw in this argument, then you might be able to see where this paper is going.

Euler covers his second "fact" in Proposition 2, which Euler states clearly, rather than briefly:

> Proposition 2: Two straight lines can cut each other in 1 point;
> Two conics can cut each other in 4 points.
> Two cubics can cut each other in 9 points.
> Two quartics can cut each other in 16 points.
> etc.
> A curve of order m can be cut by
> a line in m points,
> a conic in $2m$ points,
> a cubic in $3m$ points,
> and in general, an nth order curve in nm points.

He confesses that "The proof of this proposition is not so easy, and I will speak of it in more detail later in this discourse." Since the proof of this depends so much on the Fundamental Theorem of Algebra, and Euler can't prove that yet, he doesn't actually try to *prove* Proposition 2, but he does give several examples.

The problem is that Proposition 1 and Proposition 2 seem to contradict each other. Proposition 1 says, for example that nine points determine a cubic, and Proposition 2 says that two cubics can intersect each other at 3 times 3 is 9 points. So, pick two cubics that intersect in nine points. Use those nine points to determine a cubic. Which of the two cubics do the nine points determine?

As a concrete example, we could take the two cubics $x(x-1)(x+1)=0$ and $y(y-1)(y+1)=0$. The first gives three vertical lines, with x-intercepts –1, 0 and 1, and the second gives three horizontal lines. They intersect in the nine points with coordinates involving only –1, 0 and 1. Those nine points seem able to determine (at least) two different curves, $x(x-1)(x+1)=0$ and $y(y-1)(y+1)=0$.

The quandary gets even worse with quartics, where Proposition 1 tells us that 14 points determine a quartic, but Proposition 2 tells us that two quartics can intersect in 16 points, so 16 points seem to be able to determine *two* quartics. More constraints should give us *fewer* solutions, not more.

In the face of these apparent contradictions, Euler tells us that

> It is absolutely necessarily that either
> a. One of the two general propositions is false, or
> b. The consequences we draw are not justified.

Euler is working hard to build the dramatic tension here. Finally, without the modern tools of linear algebra, he gives a series of examples that work towards something like the rank of a system of linear equations. His first example is of two lines:

$$\begin{aligned} 3x - 2y &= 5 \\ 4y &= 6x - 10 \end{aligned}$$

Here, two lines intersect, not at one point, but at infinitely many points. That might not be what Proposition 2 meant, since there are not really two lines here, just one line written two different ways. But he expands on the idea. He gives a three-variable example,

$$\begin{aligned} 2x - 3y + 5z &= 8 \\ 3x + 5y + 7z &= 9 \\ x - y + 3z &= 7 \end{aligned}$$

Since the sum of equations 2 and 3 is the double of equation 1, these three equations have infinitely many solutions, not just one.

Euler further gives a four-variable example, and tells us that this can happen with any number of equations. He then uses the ideas to construct sets of points that correspond to more curves than Proposition 1 says they should. Let's look at what he does with a conic, or second order equation.

A conic is given by second order equation like $\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \zeta = 0$. Now, pick five points, denoted by Roman numerals I to V, such that

I = (0, 0) II = (a, 0) III = (0, b) IV = (c, d) and V = (e, f).

Euler wants to find values of a, b, c, d and e so that these five points determine more than one conic curve. These five points give us five simultaneous equations:

$$\text{I.} \quad \begin{aligned} \zeta &= 0 \\ \alpha a^2 + \delta a + \zeta &= 0 \\ \gamma b^2 + \varepsilon b + \zeta &= 0 \\ \alpha c^2 + \beta cd + \gamma d^2 + \delta c + \varepsilon d + \zeta &= 0 \\ \alpha e^2 + \beta ef + \gamma f^2 + \delta e + \varepsilon f + \zeta &= 0 \end{aligned}$$

From the first two equations, we get

$$\begin{aligned} \zeta &= 0 \\ \delta &= -\alpha a \\ \varepsilon &= -\gamma b \end{aligned}$$

Substituting these into the last two equations, we get

$$a c^2 + b cd + g d^2 - a ae - g bd = 0$$
$$a e^2 + b ef + g f^2 - a ae - g bf = 0$$

These two equations will determine the curve (two equations in three unknowns, α, β and γ, but we only need ratios), *unless* the two curves are equivalent. To find when they are equivalent, Euler solves both equations for β and sets them equal, to get

$$b = \frac{ac(a-c)+gd(b-d)}{cd} = \frac{ae(a-e)+gf(b-f)}{ef}$$

A bit of algebra shows that, for these two expressions to be equal, it is sufficient that

$$a = \frac{cf-de}{f-d}, \text{ and}$$
$$b = \frac{de-cf}{e-c}$$

So, if we pick our five points I, II, III, IV and V so that these last two relations hold, then we get five equations that have not just one, but infinitely many solutions, since the last two equations turn out to be equivalent.

This points the way to resolving the paradox. It is more complicated, but the same thing can be done with nine points and a cubic curve given by the equation

$$a x^3 + b x^2 y + g xy^2 + d y^3 + e x^2 + z xy + h y^2 + q x + i y + k = 0.$$

We pick nine points, Euler would call them I to IX. We can assume a few of our coefficients are zero, but we are still left with a pretty nasty 9x9 system of equations. To add extra complication, there are a great many ways that the system might have infinitely many solutions. For example, equation VIII might be equivalent to equation IX, or equation VII might be equivalent to the sum of equations V and VI. Euler admits that "It is difficult to formulate a general case" that describes all the different ways the equation might have infinitely many solutions.

That would take linear algebra, a subject Euler helped make necessary, and which Cramer, with his Rule, helped discover.

So, there is no paradox, the scene is set for the invention of linear algebra, and mathematics survives another crisis.

References:

[O'C] O'Connor, J. J., and E. F. Robertson, "Gabriel Cramer", The MacTutor History of Mathematics archive, http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Cramer.html.

[E147] Euler, Leonhard, Sur une contradiction apparente dans la doctrine des lignes courbes, *Mémoires de l'Academie des Sciences de Berlin*, **4** 1750, pp. 219-233, reprinted in *Opera Omnia* Series I vol 26 pp. 33-45. Available through The Euler Archive at , www.EulerArchive.org

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How Euler Did It is updated each month.