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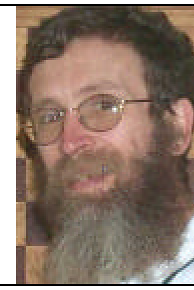
How Euler Did It

by Ed Sandifer



V, E and F, Part 2

July, 2004

Last month, we began to examine Euler's two papers on the general properties of polyhedra, often cited as the pioneering work on the subject and one of the first contributions to the field of topology. Euler had written what is usually regarded as the first topology paper in 1736, when he wrote of the bridges of Königsburg.

The Euler wrote his first paper on this subject, "Elements of the doctrine of solids," in 1750. In that paper, he claimed two main results:

> Proposition 4: In any solid enclosed by planes, the sum of the number of solid angles the number of faces exceeds the number of edges by 2.

> Proposition 9: The sum of all the plane angles which occur on the outside of a given solid is equal to eight less than four right angles for each solid angle.

In E-230, Euler seems to think that these two results are of about equal importance. He gives a proof of the second of these, using the first one, but he says that he is unable to find a proof of Proposition 4, the familiar Euler Formula, that $V - E + F = 2$.

Last month, we listed four "half-truths" about Euler and the Euler Formula, and we set about to find what truth there was in these half-truths. When we finished reading E-230, we left it like this:

1. Euler got it wrong, because he thought his formula applies to *all* polyhedra.
   This still seems half-true. Sometimes, as in Proposition 1, Euler claims his theorem is true for *any* solid. When he does that, he sometimes over-reaches. However, for his ground-breaking claims, as in Proposition 4, that $V + F = E + 2$, Euler specifically states the condition that the polyhedron be *enclosed by planes*, a condition that is sufficient to make his claims true.

2. Euler couldn't provide a proof for his formula.
   Indeed, in E-230, Euler says that he cannot give a good proof. Let's wait for E-231, though.

3. Euler gave a proof, but the proof was wrong.
   Let's wait for E-231 for this, too.

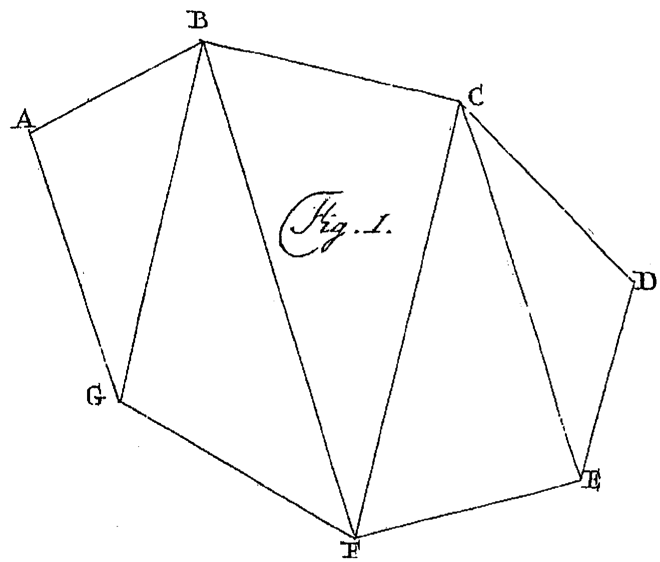
4. It shouldn't be Euler's Formula at all, since Descartes did it first.
   More on this later. It is a remarkable story.

   Now, let us turn to the sequel, and see how these half-truths stand up to the rest of the story.

E-231 – Proof of some of the properties of solid bodies enclosed by planes

Euler tells us right in the title of E-231 that he has a proof of his formula, and that he is not considering general solid bodies, like tori or spheres, but only those "enclosed by planes." This condition is sufficient to make $V+F=E+2$ true.

Euler opens the paper by reviewing what he considers to be the two main results of E-230, that $V+F=E+2$ and the one about the sums of the plane angles in a polyhedron. Then he begins preparations for the proof his title promises. He plans to prove $V+F=E+2$ by a kind of reduction argument. Today we would structure the proof as a mathematical induction, but that was not yet a popular style of proof in 1751. To prepare his readers for his proof technique, he first demonstrates how it works in two dimensions, to show that the angles in a polygon with A angles sum to $2A-4$ right angles. Suppose, for example, *ABCDEFG* is the polygon in Euler's Fig. 1 at the right, implicitly assumed to be convex. It is described by its vertices. Euler dissects the polygon into triangles by drawing diagonals *GB, BF, FC, CE*.



Euler goes into some detail to say that removing triangle *CDE* leaves a polygon with $A-1$ angles, and it degreases its angle sum by two right angles. Eventually, after n steps, for some n, we get $A-n=3$ and are left only with triangle *ABG*, for which the angle sum is two right angles. Therefore, the number of right angles in the original polygon was $2n+2$, or, equivalently, $2A-4$.

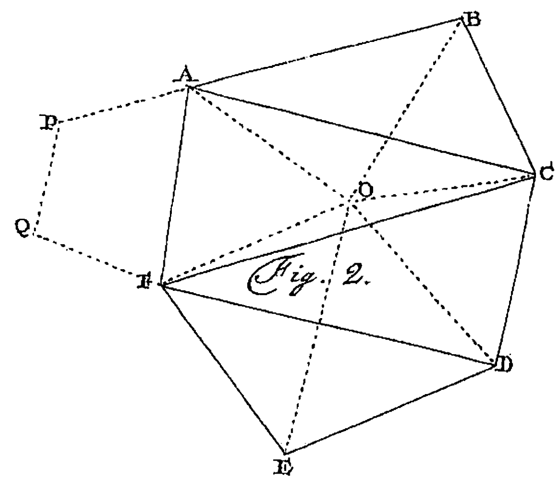
There are easier ways to do this, but Euler wanted to prepare us for his technique in three dimensions.

Next, Euler looks for a three dimensional analogy to the triangulation he used in two dimensions. He proposes choosing any point on the interior of the polyhedron, and extending edges to each vertex of the polyhedron. This leads to a decomposition of the polyhedron into pyramids, with the faces of the original polyhedron as the bases of the pyramids. He tells us that this technique does not work, but he does not tell us why he thinks that. I think a modern mathematician could make it work in a way that Euler would find convincing

Thinking that this approach wouldn't work, Euler turned to a different kind of reduction plan. He decided to remove vertices. He makes some mistakes, and he makes them all in his lemma:

Proposition 1: Given any solid enclosed by planes, then a given solid angle being cut off, in the solid that remains, the number of solid angles will be one less.

His plan, illustrated in his Fig. 2 at the right, is to remove the vertex labeled *O*. Vertex *O* is connected to vertices *A*, *B*, *C*, *D*, *E* and *F*, so when he removes *O*, he also takes a number of triangular pyramids, *OABC*, *OACF*, *OCDF*, and *ODEF*. (Notice that points *P* and *Q* shown in Fig. 2 are not used in Proposition 1. Euler recycles Fig. 2 in Proposition 2, where he does use them.) He then shows that the operation of removing vertex *O* along with all of these triangular pyramids preserves the relationship among *V*, *E* and *F*. He concludes that if he keeps removing vertices like this, eventually he'll have only four vertices left, that is, he'll have a triangular pyramid. Since $V+F=E+2$ for triangular pyramids, he concludes that $V+F=E+2$ in the original polyhedron as well.



Euler argues like this. If we remove vertex *O*, then *V* decreases by 1. Suppose that *k* is the number of faces that meet at *O*. Then *k* is also the number of edges that meet at *O*, as well as the number of sides to the (perhaps non-planar) polygon represented by *ABCDEF* in Fig. 2. Thus, the polygon triangulates to give $k - 2$ polygons, and that triangulation requires the introduction of $k - 3$ edges. Hence, removing *O* takes away one vertex (*O*), *k* edges (*OA*, *OB*, etc.) and *k* faces (*OAB*, *OBC*, *OCD*, etc.). However, it also adds the $k - 2$ faces of the triangulation and the $k - 3$ edges that went into the triangulation. So, in the new polyhedron, *V* is replaced with $V - 1$, *E* is replaced by $E - k + (k - 3) = E - 3$, and *F* is replaced by $F - k + (k - 2) = F - 2$. Thus $V - E + F$ is replaced by $V - 1 - (E - 3) + F - 2 = V - E + F$, that is, $V - E + F$ does not change. So, of $V - E + F = 2$ after all the reductions, that is, for a triangular pyramid, then $V - E + F = 2$ for the original polygon.

That would be a beautiful proof, were it not for a subtle but serious flaw. An example is illustrated in Fig. 3. If the initial polyhedron is "bounded by planes," then the polyhedron resulting after removing the vertex *O* ns not necessary still bounded by planes. So, we might not be able to repeat the process of removing vertices. In the Fig. 3, for example, removing the vertex *O* according to Euler's recipe leaves two tetrahedra, *AEBD* and *CFBD*, joined along the common edge *BD*. Euler would not accept this object as a polyhedron.

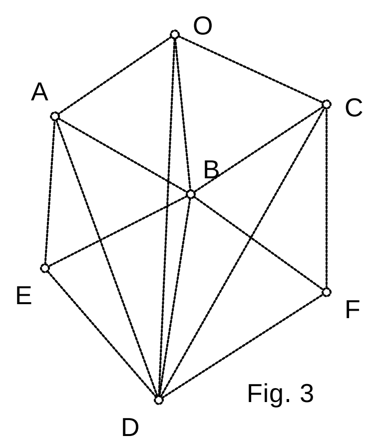


Fig. 3

Polyhedra that result from removing a vertex can have other unexpected properties that can make repairing this proof more difficult than we might expect, and, even if the proof can be repaired, the fact remains that the proof given by Euler is flawed and incomplete. Still, the theorem is true.

The End of the Story

Let's review our list of half-truths.

1. Euler got it wrong, because he thought his formula applies to *all* polyhedra.

Not true. Euler may have *thought* it applied to all polyheda, but he only *claimed* that it applied to "polyhedra bounded by planes," that is, convex polyhedra, and it does apply to them.

2. Euler couldn't provide a proof for his formula.
   Half true. Euler couldn't give a proof in his first paper, E-230, and he said so, but a year later, in E-231, he gives a convincing proof.

3. Euler gave a proof, but the proof was wrong.
   True. The proof Euler gives in E-231, though convincing, is incorrect.

4. It shouldn't be Euler's Formula at all, since Descartes did it first.
   We think this is half-true, but it contains a grain of truth. To know why, we have to tell a story. We rely mostly on Crowell [C, pp. 181-189] for our facts.

In 1649, the year before he died, Descartes went to Sweden to tutor Princess Christina in philosophy. When he died, his possessions were sent back to France, but, just as they arrived in Paris, the box of his manuscripts fell into the river. They were mostly rescued, and dried, and some were available to scholars at the time, including Leibniz. Leibniz hand-copied several of the manuscripts, including one of sixteen-pages titled *Progymnasmata de solidorum elementis,* [D] or "Exercises in the elements of solids." The original subsequently vanished, and Leibniz' copy disappeared into his papers until it was rediscovered in 1860. Thus, for over 200 years, only Leibniz knew of Descartes' work on polyhedra, and he doesn't seem to have told anybody. Nobody in Euler's time would have known of it.

The *Progymnasmata* is the first known study of general polyhedra. Even though it is not well-known, it is the first. That distinction is often credited to Euler.

So, what did Descartes do? He studied something closely related to Euler's formula for the sum of the plane angles of a polyhedron. In Descartes' time, people had a concept of a solid angle called the *deficiency*. The *deficiency* of a solid angle is the amount by which the sum of the plane angles at the solid angle fall short of four right angles. In the case, for example, of a solid right angle, formed by three right angles, the deficiency will be one right angle. For a cube, which contains eight solid right angles, the total deficiency is eight right angles. Descartes' main result is that this always happens:

Theorem: The sum of the deficiencies of the solid angles of a polyhedron is always eight right angles.

It is an almost trivial step from this to Euler's theorem, that the sum of the plane angles is four times the number of solid angles, less eight right angles, that is $4V - 8$ right angles.

Descartes' other interesting result is more subtly related, but still remotely equivalent to $V - E + F = 2$. Descartes writes:

> Dato aggregato ex omnibus angulis planis et numero facierum, numerum angulorum planorum invenire: Ducatur numerus facierum per 4, et productum addatur aggregato ex omnibus angulis planis, et totius media pars erit numeris angulorum planorum.
>
> Given the sum of all the plane angles and the number of faces, to find the number of plane angles: The number of faces is multiplied

by 4, and to the product is added the sum of all the plane angles, and the half part of this total will be the number of plane angles.

It is easy, but not obvious, to transform this rule into Euler's $V - E + F = 2$, as follows.

Let Σ denote the sum of all the plane angles in the solid, and let V be the number of vertices, what Descartes and Euler both call "solid angles." Then Descartes has just told us that $4V - \Sigma = 8$ right angles. Further, let F be the number of faces and P the number of plane angles.

It is easy for us to see that the number of plane angles is twice the number of edges, that is, $P = 2E$.

Now, Descartes tells us that given Σ and F, we are to find P. He uses words amounting to the formula

$$\frac{4F + \Sigma}{2} = P$$

Substituting $P = 2E$ and $\Sigma = 4V - 8$, we get

$$\frac{4F + 4V - 8}{2} = 2E$$

which is easy to transform into

$$V - E + F = 2.$$

So, did Descartes find the Euler formula? It would make a good debate.

References:

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Figures were downloaded from The Euler Archive, www.EulerArchive.org.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 32 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.