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How Euler Did It

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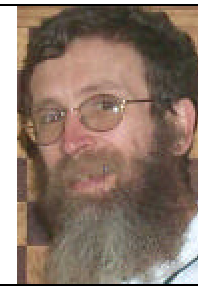

V, E and F, Part 1

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About 15 years ago, the *Mathematical Intelligencer* polled its readers to choose the ten most beautiful theorems in mathematics. The top two were results of Euler. The "Euler Identity," $e^{p i} = -1$ ranked at the top of the list (more about this in a future column), with the "Euler Formula" $V - E + F = 2$ right below it. This month's and next month's columns are about the Euler Formula.

Euler's formula tells us that if we have a polyhedron that satisfies certain conditions, and if the polyhedron has V vertices, E edges and F faces, then $V - E + F = 2$.

This is, indeed, a beautiful and popular result. Elementary school teachers use the result to try to lead their students to discover mathematical truths, and professional mathematicians use its generalizations in their research in fields such as algebraic topology and differential geometry.

Unfortunately, those "certain conditions" we glossed over can get a little bit nasty. We have to exclude, for example, polyhedra with holes in them, like donuts, and polyhedra with disconnected interiors, like two cubes joined at a vertex. It can get discouraging, as Imre Lakatos showed so brilliantly in his 1976 book *Proofs and Refutations* [L]. After reading that book and seeing so many challenging counterexamples, we are tempted to despair and say that the formula is true just for those polyhedra for which it is true.

Euler could not have known most of the difficult examples that Lakatos uses. Since most people have not read his original papers from 1750 and 1751, written in Latin, but many have read Lakatos and other modern sources, some half-truths have arisen about what Euler did and what he proved in those two papers. Among those half-truths are

1. Euler got it wrong, because he thought his formula applies to *all* polyhedra.
2. Euler couldn't provide a proof for his formula.
3. Euler gave a proof, but the proof was wrong.
4. It shouldn't be Euler's Formula at all, since Descartes did it first.

In fact, half of these statements are more or less half true. Our purpose here is to describe what Euler did and how he did it, and to try to figure out which parts of each of these "half-truths" are true.

Euler in the late 1740's and early 1750's enjoyed some of the richest creative years of any mathematician or scientist ever. Almost single-handedly he filled the pages of two of the world's most important scientific journals the *Mémoires* of the Berlin Academy and the *Novi commentarii* of the St. Petersburg Academy. In 1750, for example, he published 35 papers, and another 20 in 1751. In 1750, he turned some of his attention to the properties of solids, a subject he called "stereometry." The result was his first paper on the subject, number 230 on the Eneström index [E-230], titled "Elementa doctrinae solidorum," or "Elements of the doctrine of solids." A year later he wrote a shorter sequel [E-231], "Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusis sunt praedita," or "Proof of some of the properties of solid bodies enclosed by planes." These two papers were published back-to-back in the 1752/53 volume of the *Novi commentarii* of the St. Petersburg Academy, which, subject to the typical publication delay of the times, appeared in print in 1758.

E-230 – Elements of the doctrine of solids

Euler begins E-230 with an eloquent description of a grand plan to put the geometry of solids on the same elegant foundations as Euclid did for plane geometry. We get the impression that he hoped to write a great number of papers on the new subject of *stereometry*, though, in fact, he only wrote these two.

Then, by analogy with polygons, which consist of points and lines, he tells us that the solids he wants to study consist of points, lines and planes. The points are each solid angles, formed where three or more planes come together. He calls them *anguli solidi* and denotes their number by S. We will use the English language tradition and call them vertices, and denote them by V. The faces he calls *hedra*, denoted by H. We will use F.

The lines are a problem for Euler, though. He tells us that they do not have a proper name, so he decides to call them *acies*, which translates as keenness, edge, penetration, insight, or battle line. What we now call an edge hadn't been named yet. Where he used A to count them, we will use E.

Most solids don't have names. Euler invests some time in developing a nomenclature, that didn't catch on, based on the numbers of vertices and faces. For example, a triangular prism, with six vertices and five sides, he called a *pentaedrum hexagonum*, or "five-faced hexagon."

Eventually, seven pages into the 30-page paper, Euler gets to some theorems. He wants to count the number of "plane angles" in a solid. It is not a quantity we still use very often, and Euler never assigns the number a variable name, so let's call it P. For example, a cube has six 4-sided faces, making a total of 24 plane angles, so for a cube, $P = 6 \cdot 4 = 24$. Now, Euler's Proposition 1 is

> *Propositio 1: In quovis solido numerus omnium acierum est semissis numeri omnium angulorum planorum, qui in cunctis hedris ambitum eius constituentibus reperiuntur.*
>
> Proposition 1: In any sold, the number of edges is half of the number of plane angles which are located in the corners of the faces.

As a formula, Euler is telling us that $E = P/2$. Euler adds corollaries to this proposition. In the days before subscripts, he let a be the number of triangular faces on the solid, b the number of quadrilaterals, c the number of pentagonal faces, etc. Then we have two more equations:

$$F = a + b + c + d + e + \text{etc.}$$

$$P = \frac{3a + 4b + 5c + 6d + 7e + \text{etc.}}{2}$$

Euler continues, proving two propositions that we usually prove as corollaries of the Euler Formula itself, that $2E \geq 3F$ and $2E \geq 3V$.

Now Euler gets to his first big theorem:

> *Propositio 4: In omni solido hedris planis incluso aggretatum ex numero angulorum solidorum et ex numero hedrarum binario excedit numerum acierum*

> Proposition 4: In any solid enclosed by planes, the sum of the number of solid angles the number of faces exceeds the number of edges by 2.

As a formula, this is $V + F = E + 2$. Euler never writes it as $V - E + F = 2$.

Then Euler seems to begin a proof of the theorem! But the proof begins with an apology: "I have not been able to find a firm proof of this theorem." Instead, he works through a series of progressively more complicated and general examples, and ending with a check of the five Platonic solids. But he admits that, however convincing these examples may be, they do not make a proof.

Assuming that the proposition is true, Euler does prove a result that is essential to studies of the Four Color Theorem:

> Proposition 7: There cannot exist a solid all of whose faces have six or more sides, nor can there exist a solid all of whose solid angles are formed by six or more plane angles.

Finally, he turns to a theorem that he regards as important as $V - E + F = 2$, but which is practically unknown today. He gives two versions of the same result:

> Proposition 8: The sum of all the plane angles which are in a given solid is equal to four right angles for each unit by which the number of edges exceeds the number of solid angles.

> Proposition 9: The sum of all the plane angles which occur on the outside of a given solid is equal to eight less than four right angles for each solid angle.

If we let *S* be the sum of all the plane angles, measured as multiples of 90° (radians hadn't been invented yet), then Proposition 8 tells us that $S = 4E - 4V$ right angles, and Proposition 9 says that $S = 4V - 8$ right angles.

Euler gives a correct proof of Proposition 8, then uses the Euler Formula to prove Proposition 9 as a consequence.

This brings us to the end of E-230, Euler's first paper, written in 1750. We can pause briefly to see how Euler is doing with regard to the four "half-truths" listed above

1. Euler got it wrong, because he thought his formula applies to *all* polyhedra.
   This still seems half-true. Sometimes, as in Proposition 1, Euler claims his theorem is true for *any* solid. When he does that, he sometimes over-reaches. However, for his ground-breaking claims, as in Proposition 4, that $V+F=E+2$, Euler specifically states the condition that the polyhedron be *enclosed by planes*, a condition that is sufficient to make his claims true.

2. Euler couldn't provide a proof for his formula.
   Indeed, in E-230, Euler says that he cannot give a good proof. Let's wait for E-231, though.

3. Euler gave a proof, but the proof was wrong.
   Let's wait for E-231 for this, too.

4. It shouldn't be Euler's Formula at all, since Descartes did it first.
   More on this later. It is a remarkable story.

This story is getting long. Rather than burden the reader with a double-length column, let's think about this for a month before we turn to E-231, Euler's second paper on polyhedra.

References:

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[E231] Euler, Leonhard, Demonstratio nonnullarum insignum proprietatum quibus solida hedris planis inclusa sunt praedita, *Novi commentarii academiae scientiarum Petropolitanae* **4** (1752/3) 1758, p. 140-160, reprinted in *Opera Omnia* Series I vol 26 p. 94-108.

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Figures were downloaded from The Euler Archive, www.EulerArchive.org.

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How Euler Did It is updated each month.