



Mixed Partial Derivatives

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One of the first things we learn in Calculus III, multivariable calculus, is that mixed partial derivatives are equal. That is, for most familiar functions of two variables, say $f(x, y)$, it doesn't matter whether you take partial derivatives first with respect to x , then with respect to y , or if you do it the other way around. In symbols, this says $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, or $f_{xy} = f_{yx}$.

Almost the next thing we learn is that there are a few conditions of continuity that our function $f(x, y)$ must satisfy to assure this equality. We learn a number of special counterexamples, which, I for one, remembered for the test, but then forgot until I had to teach Calculus III myself.

Two hundred and seventy years ago, the fact about partial derivatives was unknown. The very idea functions was new. People used equations of two variables to describe curves, and of three variables to describe surfaces, but they hadn't made the transition from equations to functions. Surfaces were additionally difficult to deal with because three-dimensional coordinate systems were new and people were not yet comfortable with them.

Somehow, in the midst of this confusion, Euler was able to discover the fact that mixed partial derivatives are equal. Since Euler did not know any of the functions that could have served as counterexamples, we should not begrudge it that he did not also discover the continuity conditions.

We will explore Euler's discovery by looking at three questions. First, without the tools of surfaces, functions and three-dimensional coordinate systems, how could Euler make this discovery? Second, exactly what did he discover, anyway? And finally, what kind of proof or evidence did he offer to make people believe it was true?

The answer to the first question is a little surprising. Euler wasn't thinking about surfaces, functions or three-dimensional coordinate systems when he wrote this paper in 1734. The paper whose title in English is "On an infinity of curves of a given kind, or a method of finding equations for an infinity of curves of a given kind" is number 44 on Eneström's index of Euler's works. The title is as awkward in its original Latin as it is in English. As the title suggests, the paper is about families of curves. The title does not suggest that Euler means to study the differential equations satisfied by a given family of curves.

Euler begins his paper with a discussion of what we would call a *parameter*, but Euler calls a *modulus*. It is a relatively rare example of a term that Euler used that was not adopted by the rest of the mathematical community. He uses an example, $y^2 = ax$, which he interprets as describing infinitely many parabolic curves, one for each value of a , and all with the same axis and vertex. He intends to examine how the curves change as the value of a changes. So we see that Euler was thinking about curves, parameters and two-dimensional coordinate systems, not surfaces, functions and three-dimensional coordinate systems.

What was he thinking about? At the time, people didn't say they were "solving" a differential equation. Instead, they said they were "integrating" or "constructing" it, so it was natural for a study of differential equations to begin with integration. Euler asks us to consider $z = \int P dx$, where P involves a , z and x . Then $dz = P dx$, in which expression a is to be considered as a constant. Euler works hard to explain that if a is considered a variable, then this last expression could be differentiated with respect to a , and also that if $dz = P dx$ is integrated, then the resulting expression might involve a function of a . All this leads to a conclusion that seems paradoxical when we first see it, that if a is a constant, then

$$dz = P dx$$

but if a is considered as a variable, then

$$dz = P dx + Q da.$$

Euler moves on to state a theorem: "If a quantity A composed of two variables t and u is differentiated first holding t constant, and then that differential is differentiated holding u constant and letting t be a variable, then the same result will occur if the order is reversed and A is first differentiated holding u constant and then that differential is differentiated holding t constant and letting u be a variable."

We recognize this as claiming that mixed partial derivatives are equal, regardless of the order of the differentiation. Today, this is the second thing we learn about partial derivatives when we encounter them in calculus class. There, we also learn about some continuity conditions that Euler does not yet know about.

The awkward wording and the lack of notation are dictated by Euler's times. He writes about differentials and not derivatives, so the very idea of partial derivatives and second order partial derivatives is more difficult to discuss. Our modern notations for partial derivatives have evolved over many years specifically so that it is easy to use them to write facts like this. Compare $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ or $f_{xy} = f_{yx}$ to the tools Euler had to make this same statement.

Our third question was about Euler's proof. For this, we can translate Euler himself:

Suppose A is a function of t and u . From A we get B if, in place of t in A we substitute $t + dt$; and we get C if in place of u in A we substitute $u + du$. If we simultaneously substitute $t + dt$ and $u + du$, we change A into D . From a different point of view, we could get D by substituting $u + du$ for u in B or by substituting $t + dt$ for t in C . This said, if the differential of A is taken, holding t constant, it will produce $C - A$. If in $C - A$ we put $t + dt$ in place of t , it will produce $D - B$, the differential of which will be

$$D - B - C + A.$$

Now, doing things in the other order, if $t + dt$ is put into A in place of t , then B is produced, and then the differential of A , taking t to be the variable, will be $B - A$. Putting $u + du$ in place of u in this differential will give

$$D - B - C + A,$$

which is equal to the differential found in the previous operations.

Q. E. D.

All of this fills only three pages of Euler's 20 page paper. The rest of the paper is concerned with some now-forgotten questions of how an integral like $z = \int P dx$ depends on a , if P is a function of x and also involves a parameter a . He takes the differential, $dz = P dx + Q da$, and Euler wants to understand what Q is. Euler takes more differentials, $dP = Adx + Bda$ and $dQ = Cdx + Dda$. Because of Euler's result about mixed partial derivatives, he knows that $C = B$, so $dQ = Bdx + Dda$, and finally, $Q = \int Bdx$. This part of the paper goes on and on, as Euler considers different forms for P and adds more and more complications.

It is a clear case for which the tool is more important than the problem.

There are two other minor features of this paper. The first involves e , the base for natural logarithms. Euler pioneered the use of standard symbols for such constants. He was also responsible for our universal use of the symbol π . Euler's first "official" use of e was in his two volume *Mechanica*, published in 1736. This paper was *written* in 1734, but not published until 1740. Publication delay cost it first place.

Second, some sharp-eyed readers may have noticed that this paper occupies pages 174-189 and 180-183 of its volume of the *Commentarii*. It's not our typographical error. A typesetter in 1740 got confused as he set the page numbers for this volume, and after page 189 he put another page 180.

Reference:

[E] Euler, Leonhard, De infinitis curvis eiusdem generic seu methodus inveniendi aequationes pro infinitis curvis eiusdem generic, *Commentarii academiae scientiarum Petropolitanae* 7 (1734/5) 1740, p. 174-189, 180-183, reprinted in *Opera Omnia* Series I vol 22 pp. 36-56.

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