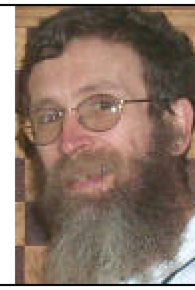
# How Euler Did It

by Ed Sandifer



## Beyond Isosceles Triangles

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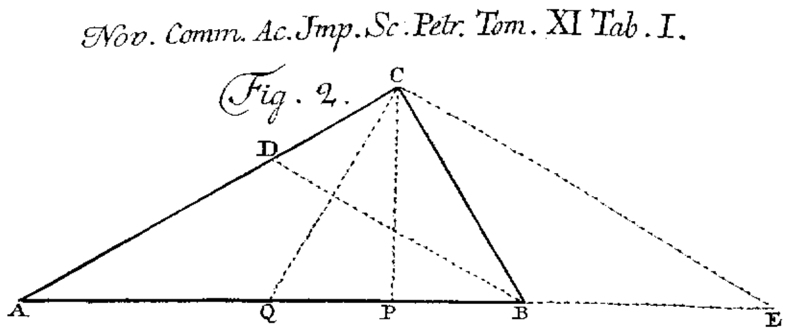
We know lots about triangles for which $\angle A = \angle B$. Such triangles are isosceles, and we have known at least since Euclid that $\angle A = \angle B$ exactly when $a = b$. In 1765, Euler studied a generalization of this situation. What happens if $\angle B$ is some mu ltiple of $\angle A$?

In 1765, Euler was about to return to St. Petersburg after 25 years in Berlin. His last years in Berlin had been rather unhappy, and he would glad to be back in Russia, where the new Empress, Catherine II, also known as Catherine the Great, showered him with attention and honors. While he was in Berlin, he continued to edit, the *Novi commentarii academiae scientiarum imperialis petropolitanae*, the main journal of the St. Petersburg Academy. He also published mu ch of his most important work there. In 1766, Euler left Berlin, and was back in St. Petersburg by the time the volume was actually published in 1767. In the 1765 volume, Euler wrote no less than ten papers on an amazingly diverse range of topics. One was about the nature of discontinuous functions, two about the three body problem in celestial mechanics, one about the so-called Pell equation in number theory, one about friction in gears, and one was the paper in which he discovered the well-known Euler Line. Nestled among these others, numbered E 324 in Eneström's index of Euler's work, was a 35 page paper, "Proprietates triangulorum quorum anguli certam inter se rationem tenent" (Properties of triangles for which certain angles have a ratio between themselves.)

Euler let ABC be a triangle. As usual, he denotes the side opposite angle A by $a$, opposite B by $b$ and opposite C by $c$. To set up his problem, Euler supposes that the ratio between the angles is given by $\angle A : \angle B :: m : n$ and that $n = 1$. He considers various values of $m$. First, he reminds us that if $m = 1$, then we are talking about isosceles triangles, so $a = b$. To make this fit certain patterns later in the paper, he chooses to write this as $b - a = 0$. He continues to the cas e $m = 2$. By the end of the paper, he will have studied as far as $m = 13$.

In case $m = 2$, Euler poses the problem "to investigate the relations that arise." He gives us Figure 2 below. In Euler's day, all the figures in the book were printed together on pages bound at the back of the book, and numbered consecutively. Figure 1 was part of another article on discontinuous functions.

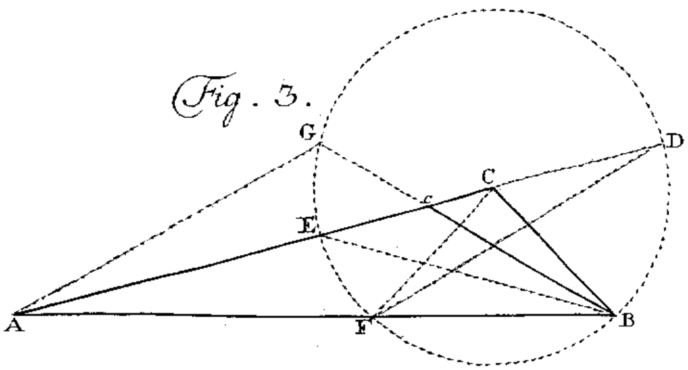



In Figure 2, Euler takes $\angle B = 2\angle A$ and draws BD bisecting $\angle B$. This makes $\angle A = \angle ABD = \angle DBC$ so triangle ADB is isosceles and triangles ABC and BDC are similar. He writes the similarities as $AC : BC = AB : BD = BC : CD$ so $b : a = c : \frac{ac}{b} = a : \frac{aa}{b}$ and so $BD = \frac{ac}{b}$, $CD = \frac{aa}{b}$ and $AD = b - \frac{aa}{b}$. Note that to make typesetting easier, in Euler's day they usually wrote $aa$ where we would write $a^2$.

Euler has expressions for AD and for BD, and he knows they are equal since ADB is isosceles, and it follows immediately that $ac = bb - aa$. This is Euler's main result for triangles where one angle is the double of another.

Readers will note that Euler has only used about half of his diagram so far. He hasn't used the other points P, Q and E. He draws CP an altitude of triangle ABC, and then locates Q and E so that QP = PB and AP = PE. Then he uses this diagram to show that if the sides of a triangle satisfy the relation $ac = bb - aa$, then $\angle B = 2\angle A$. We will leave it to the reader to work that out.

Euler goes on to consider the case $m = 3$, where $\angle B = 3\angle A$. He uses Figure 3 below, in which he adds a new point $c$ such that $\angle CBc = \angle A$. It doesn't bother Euler that c denotes both a point on side AC and the length of side AB because it is always clear what he means from the context. Since he already knows that $\angle B = 3\angle A$, he doesn't have to trisect $\angle B$ to do this, though the idea of finding a way to trisect angles probably occurred to him.




This leaves $\angle ABc = 2\angle A$, so his earlier results apply to triangle AB*c*. He writes the lengths of the sides of this triangle using Greek letters αβ and γ, so that $bb - aa - ag = 0$.

Also, since $\angle CBc = \angle A$, triangles ACB and BC*c* are similar, and we have the ratios $b : a = c : \frac{ac}{b} = a : \frac{aa}{b}$ and so $Bc = \frac{ac}{b}$, $Cc = \frac{aa}{b}$ and $Ac = \frac{bb - aa}{b}$.

Comparing the sides that triangles AB*c* and ABC have in common, we know that $g = c$, $b = \frac{bb - aa}{b}$ and $a = \frac{ac}{b}$. Putting all this together and clearing the denominator, we get $(bb - aa)^2 - acc(a + b) = 0$. Then factoring out $a + b$ leaves $(bb - aa)(b - a) - acc = 0$. This is Euler's main result in the case $m = 3$.

As before, the remainder of Figure 3 plays a role in showing that triangles that satisfy the condition $(bb - aa)(b - a) - acc = 0$ must also satisfy $\angle B = 3\angle A$.

Euler continues this way, up to $m = 5$, each time drawing a line B*c* to make $\angle CBc = \angle A$, and thus cutting off one triangle BC*c* similar to triangle ABC, and another triangle AB*c* for which the immediately previous results apply, that is for which $\angle CBc = (m - 1)\angle A$. He gets progressively more complicated relations among the sides, though he stops proving that triangles satisfying those relations have angles in the given ratios.

With $m = 5$, Euler notices a pattern, proves that the pattern holds, and then uses the pattern to give relations up to $m = 13$. We'll stop here, though, before we get buried in an avalanche of details. Though this certainly wasn't one of Euler's major results, it is a pleasant one, and it is a fine example of Euler's style.

Reference:


[E] Euler, Leonhard, Proprietates triangulorum, quorum anguli certam inter se rationem tenent, *Novi commentarii academiae scientiarum imperialis petropolitanae* Vol. XI, 1765 (1767), pp. 67-102, reprinted in *Opera Omnia* Series I vol 26 pp. 109-138.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