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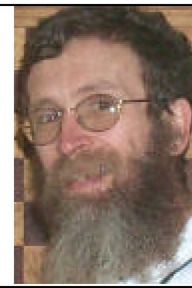
How Euler Did It

by Ed Sandifer



Venn Diagrams

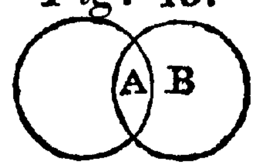
January, 2004

Almost all of us have seen and used Venn Diagrams. Though they are no longer quite as ubiquitous as they were during the heyday of "The New Math" (has it already been 35 years?), they are still a staple of Discrete Mathematics courses. Nobody calls them anything except "Venn Diagrams," and several modern sources (e. g. [R]) refer to John Venn's 1880 article [V], so it is natural to assume that Venn discovered them. That would be the end of it, unless we read Venn's article (fairly easily available via InterLibrary Loan) and see that Venn doesn't call them "Venn Diagrams." He calls them "Eulerian Circles"! We quote his opening sentences:

> "Schemes of diagrammatic representation have been so familiarly introduced into logical treatises during the last century or so, that many readers, even those who have made no professional study of logic, may be supposed to be acquainted with the general nation and object of such devices. Of these schemes one only, viz. that commonly called "Eulerian circles," has met with any general acceptance. …"

The "Schemes of diagrammatic representation" Venn writes about are ways to represent propositions by diagrams. Euler had written about them almost 120 years earlier in his *Letters of Euler on Different Subjects in Natural Philosophy*, also known as *Letters to a German Princess*. Euler wrote these *Letters*, approximately two per week, from April, 1760 to May, 1763, as lessons in elementary science for the Princess of Anhalt Dessau. They fill two volumes, over 800 pages, and cover a huge range of topics in science: light, color, gravity, astronomy, mechanics, sound, music, heat, weather, logic, magnetism, optics, and more. Though she was a German princess, the *Letters* were written in French, the language of courtly society. They were published in 1768, in part through the efforts of the French mathematicians Condorcet and Lacroix, and immediately became very popular. Soon, they were translated into all the major languages of Europe. Bill Dunham writes "*Letters to a German Princess* remains to this day one of history's finest examples of popular science." I am working from two editions, 1823 and 1833, of a 1795 English translation by Henry Hunter.

Fig. 45.

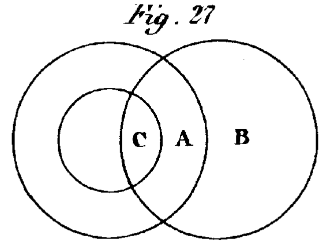


Euler uses his circles in four letters, CII to CV of Volume I, written between February 14 and February 24, 1761. Each letter has a title. CII, for example, is called "On the Perfections of a Language. Judgments and Nature of Propositions, affirmative and negative; universal or particular." The other titles are a little shorter, and mention Forms, Syllogisms and Propositions.

Syllogisms come in a variety of forms. One form is called the *affirmative particular*: Some A is B. Euler illustrates this with what is labeled "Fig. 45." Euler notices, though, that the figure does not have a unique interpretation. He offers four different interpretations of Fig. 45:

I. Some A is B.
II. Some B is A.
III. Some A is not B.
IV. Some B is not A.

Fig. 27

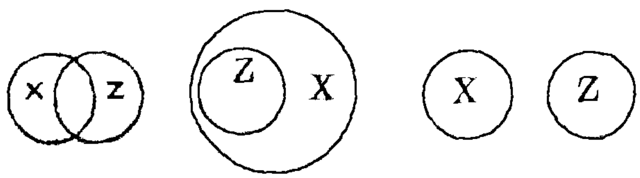


There are more than fifty such figures. Euler uses Fig. 27 to justify the following logical argument:

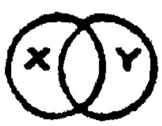
1. Some C is B.
2. All C is A.
3. Therefore, some A is B.

Euler is not thinking of sets. He is using these diagrams to explain syllogisms, propositions and logical arguments. Note also that it doesn't seem to occur to Euler that some of the regions shown in his diagram may be empty.

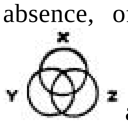
A hundred and twenty years later, John Venn returns to the same problem, how to use diagrams to represent language. Again, he is not yet interested in sets, but in logic, statements and propositions. Just as Euler notes that several statements correspond to Fig. 45, Venn notes that several diagrams may correspond to the same statement. For example, he gives the three diagrams below that correspond to the statement "Some X is not Z."



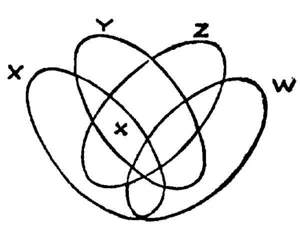
So far, it seems like Venn is just re-writing Euler using letters from the end of the alphabet instead of letters from the beginning. Some people have suggested that Venn was nothing but an arrogant usurper, taking Euler's ideas and promoting them as his own. Indeed, if Venn stops here, he probably is guilty of that. Let's look at the rest of the paper.

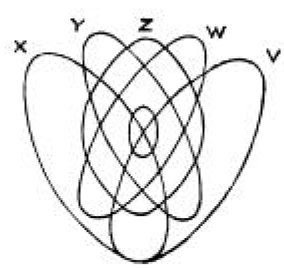


Venn notices that the simple two-circle diagram shown at the left divides the plane into four regions. He uses bars to denote "not", so the expression $X\overline{Y}$ denotes "X but not Y", and is represented by the crescent-shaped region at the left of the diagram. He continues:

> "Now conceive that we have to reckon also with the presence, and consequently with the absence, of Z. We just draw a third circle intersecting the two above, thus,  and we have the eight compartments or classes which we need. The subdivisions thus produced correspond precisely with the letter-combinations."

Venn continues with this idea. He sees that he can not draw diagrams for four propositions using circles, but he can do it using ellipses. The diagram on the left below shows how this can be done. The compartment marked with an "x" corresponds to the class $XYZ\overline{W}$.





Venn isn't able to find an arrangement of circles or ellipses that diagrams five propositions, though he does find the remarkable one shown above on the right. The regions for V, W, X and Y are ellipses, and the region for Z is shaped like a donut. People are still interested in designing Venn diagrams that represent large numbers of propositions. Frank Ruskey [R] shows a single shape that creates a Venn diagram for seven propositions.

Venn has a key idea that Euler overlooks; he can use the same diagram analyze different lists of propositions if he uses them to keep track of what compartments are empty. He refers to the diagram below when he writes

> "Ascertain what each given proposition denies, and then put some kind of mark upon the corresponding partition in the figure. The most effective means of doing this is just to shade it out. For instance, the proposition 'All X is Y' is interpreted to mean that there is no such class of things in existence as 'X that is not-Y' or $X\bar{Y}$. All, then, that we have to do is to scratch out that subdivision in the two-circle figure thus,
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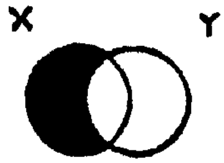
> 
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> ."

He goes on to do examples using three and four propositions, and to describe in considerable detail the process for using his diagrams. Finally, he speculates on the design of a "logical machine," based on an electric abacus that someone named Prof. Jevons built. He even gives schematic diagrams for the construction of such a machine for four propositions. It is not clear whether the machine was ever built.

Venn says a subdivision exists or does not exist, where we would probably say it is empty or non-empty. He also "scratches out" the ones that do not exist, where we usually fill in the ones that are non-empty. Still, his diagrams look considerably more like modern diagrams, 120 years after Venn, than they look like Euler's circles, 120 years before Venn. Venn may begin where Euler ends, but he does add important new ideas of his own.

Venn himself might say that if X is the mathematics Euler did, and if Y is the mathematics Venn did, then surely $X\bar{Y}$, and indeed XY, but also $\bar{X}Y$.

[D] Dunham, William, *Euler The Master of Us All*, MAA, Washington, D. C., 1999.

[E] Euler, Leonhard, *Letters of Euler on Different Subjects in Natural Philosophy*, Henry Hunter, tr., Harper, New York, 1833, reprint edition Arno Press, 1975 and Edinburgh, 1823.

[R] Ruskey, Frank, "A Survey of Venn Diagrams", *Electronic Journal of Combinatorics*, February 2, 1997 [ed March 15, 2001]

[V] Venn, John, "On the Diagrammatic and Mechanical Representation of Propositions and Reasonings", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science [Fifth Series] July 1880, 1-18.

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How Euler Did It is updated each month.