Meditations on Several Integrals whose Values can be Expressed, in Certain Cases, by Trigonometric Functions

Leonhard Euler

Translated by Eric S. Proffitt

1. Every rational function has an antiderivative expressible as a combination of rational, logarithmic and trigonometric functions. Unfortunately, in the multivariate case, these antiderivatives are often exceptionally complicated. However, when one considers certain definite integrals of rational functions, it sometimes occurs that these integrals, however complex they may be, reduce to simple families of antiderivatives, some of which merit particular attention.

In addition, there are also certain integrands which, in general, surpass all known analytic methods, and yet which nevertheless, in certain cases, have antiderivatives expressible in terms of trigonometric functions alone. In this paper I propose to consider several such integrals, as well as to examine those results derivable from them, in service of the general advancement of analysis.

2. I will begin by considering the family of definite improper integrals

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^n}. \tag{1}$$

In preface, for this family of rational integrals, we will find that the logarithmic portion of the integral vanishes, while the remaining part, that composed up from trigonometric functions, will reduce to rather simple expressions.

Letting π denote the half-circumference length of the perimeter of a circle of radius 1, so that π also denotes the measure of two right angles, we find that the values of several members of this family of integrals reduce to the following:

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \quad \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}} \quad \int_0^\infty \frac{xdx}{1+x^3} = \frac{2\pi}{3\sqrt{3}} \quad \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

$$\int_0^\infty \frac{xdx}{1+x^4} = \frac{\pi}{4} \quad \int_0^\infty \frac{x^2dx}{1+x^3} = \frac{\pi}{2\sqrt{2}} \quad \int_0^\infty \frac{dx}{1+x^6} = \frac{\pi}{3} \quad \int_0^\infty \frac{xdx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$$

$$\int_0^\infty \frac{x^2dx}{1+x^6} = \frac{\pi}{6} \quad \int_0^\infty \frac{x^3dx}{1+x^6} = \frac{\pi}{3\sqrt{3}} \quad \int_0^\infty \frac{x^4dx}{1+x^6} = \frac{\pi}{3}.$$

3. These particular examples given above seem already sufficient to reason inductively toward a more general conclusion. Since, in those cases where the denominator is $1+x^3$, the radical $\sqrt{3}$ indicates that the antiderivative contains $\sin\frac{\pi}{3}$, and for those integrands with denominator $1+x^4$, the radical $\sqrt{2}$ almost certainly comes from $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and a similar pattern is also confirmed in those cases where the denominator is $1+x^6$. These observations thus lead us to the following equalities:

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n\sin\frac{\pi}{n}}$$

and then more generally, provided that $m \leq n$,

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\sin\frac{m\pi}{n}}, \tag{2}$$

On the other hand when $m > n$ the above formulas require augmentation, since in this case the antiderivatives retain an algebraic part.

4. Our conclusion in (2) is altogether confirmed if one further bothers to evaluate the integrals

$$\int_0^\infty \frac{dx}{1+x^5}, \quad \int_0^\infty \frac{xdx}{1+x^5}, \quad \int_0^\infty \frac{x^2dx}{1+x^5}, \quad \text{etc.},$$

which yield values that leave little doubt concerning the correctness of (2). We also remark upon perfect agreement in those cases where $m = n$, since then $\sin\frac{m\pi}{n} = \sin\pi = 0$, and thus the integral is quite clearly divergent; evident also from the observation that

$$\int \frac{x^{n-1}dx}{1+x^n} = \frac{1}{n}\log(1+x^n),$$

with the right hand side clearly diverging as $x \to \infty$. The equality in (2) is also quite clearly true when $n = 2m$, since then $\sin\frac{m\pi}{n} = \sin\frac{\pi}{2} = 1$, and thus via the substitution $x^m = y$, we have

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^{2m}} = \frac{1}{m}\int_0^\infty \frac{dy}{1+y^2} = \frac{1}{m}\arctan y\Big|_0^\infty = \frac{\pi}{2m}.$$

This should be sufficient evidence to conclude that (2) is in fact true.

5. However in order to rigorously derive (2), we consider a summation formula for the indefinite integral version of (1),

$$\int \frac{x^{m-1}dx}{1+x^n} = \sum_{\substack{1\leq i\leq n \\ i \text{ odd}}} \Big[-\frac{1}{n}\cos\frac{im\pi}{n}\log(1-2x\cos\frac{i\pi}{n}+x^2) + \frac{2}{n}\sin\frac{im\pi}{n}\arctan\frac{x\sin\frac{i\pi}{n}}{1-x\cos\frac{i\pi}{n}}\Big], \tag{3}$$

provided of course that $m \leq n$, and with the additional caveat that when n is odd it is necessary to take only half the final term in the series, or alternatively in the final term to replace $\log(1+2x+x^2)$ with $\log(1+x)$.

6. Delving into several particular cases of (3):

I. For $n = m = 1$, we will have

$$\int \frac{dx}{1+x} = \log(1+x).$$

II. For $n = 2$, we will have:

$$\begin{aligned}
\text{if } m = 1: \quad & \int \frac{dx}{1+x^2} = \frac{2}{2} \sin\frac{\pi}{2} \arctan \frac{x \sin\frac{\pi}{2}}{1 - x\cos\frac{\pi}{2}}, \\
\text{if } m = 2: \quad & \int \frac{xdx}{1+x^2} = \frac{1}{2}\log(1+x^2).
\end{aligned}$$

III. For $n = 3$, we will have:

$$\begin{aligned}
\text{if } m = 1: \quad \int \frac{dx}{1+x^3} &= -\frac{1}{3}\cos\frac{\pi}{3}\log(1 - 2x\cos\frac{\pi}{3} + x^2) + \frac{2}{3}\sin\frac{\pi}{3}\arctan\frac{x\sin\frac{\pi}{3}}{1 - x\cos\frac{\pi}{3}} \\
&\quad - \frac{1}{3}\cos\frac{3\pi}{3}\log(1+x), \\
\text{if } m = 2: \quad \int \frac{xdx}{1+x^3} &= -\frac{1}{3}\cos\frac{2\pi}{3}\log(1 - 2x\cos\frac{\pi}{3} + x^2) + \frac{2}{3}\sin\frac{2\pi}{3}\arctan\frac{x\sin\frac{\pi}{3}}{1 - x\cos\frac{\pi}{3}} \\
&\quad - \frac{1}{3}\cos\frac{6\pi}{3}\log(1+x), \\
\text{if } m = 3: \quad \int \frac{x^2dx}{1+x^3} &= -\frac{3}{3}\cos\frac{\pi}{3}\log(1 - 2x\cos\frac{\pi}{3} + x^2) + \frac{2}{3}\sin\frac{3\pi}{3}\arctan\frac{x\sin\frac{\pi}{3}}{1 - x\cos\frac{\pi}{3}} \\
&\quad - \frac{1}{3}\cos\frac{9\pi}{3}\log(1+x),
\end{aligned}$$

or more simply, as a result of:

$$\cos\frac{3\pi}{3} = -1; \quad \cos\frac{2\pi}{3} = -1 \quad \text{and} \quad \sin\frac{3\pi}{3} = 0 \quad \text{and} \quad \cos\frac{\pi}{3} = \frac{1}{2},$$

we have

$$\text{if } m = 3: \quad \int \frac{x^2dx}{1+x^3} = \frac{1}{3}\log(1 - x + x^2) + \frac{1}{3}\log(1+x) = \frac{1}{3}\log(1+x^3).$$

7. In all these cases it is easy to see that upon letting $x \to \infty$, these integrals are in perfect agreement with formula (2) given above. However in order to demonstrate the correspondence in general, it is necessary to show that all the logarithmic parts necessarily vanish, while those parts made up of trigonometric expressions reduce to $\frac{\pi}{n\sin\frac{m\pi}{n}}$. For this purpose, it is necessary here to distinguish cases based on the parity of n.

Therefore let us begin by setting $n = 2k$ and letting $x \to \infty$. Now since the logarithmic terms all then behave asymptotically like $\log x^2$, it will only be necessary to show that the sum of the following progression for m even is equal to zero:

$$\cos\frac{m\pi}{2k} + \cos\frac{3m\pi}{2k} + \cos\frac{5m\pi}{2k} + \cdots + \cos\frac{(2k-5)m\pi}{2k} + \cos\frac{(2k-3)m\pi}{2k} + \cos\frac{(2k-1)m\pi}{2k}.$$

In order to abridge notation, we set $\frac{m\pi}{2k} = \varphi$, and thus it will be necessary to show that

$$\cos\varphi + \cos 3\varphi + \cos 5\varphi + \cdots + \cos(2k-1)\varphi = 0.$$

8. In order to find the sum of this progression, we let

$$S = \cos\varphi + \cos 3\varphi + \cos 5\varphi + \cdots + \cos(2k-1)\varphi,$$

and multiplying by $\sin\varphi$, given the identity that $\sin\varphi\cos\alpha\varphi = -\frac{1}{2}\sin(\alpha-1)\varphi + \frac{1}{2}\sin(\alpha+1)\varphi$, we will have

$$S\sin\varphi = \frac{1}{2}\sin 2\varphi + \frac{1}{2}\sin 4\varphi + \frac{1}{2}\sin 6\varphi + \cdots + \frac{1}{2}\sin(2k-2)\varphi + \frac{1}{2}\sin 2k\varphi - \frac{1}{2}\sin 2\varphi - \frac{1}{2}\sin 4\varphi - \frac{1}{2}\sin 6\varphi - \cdots - \frac{1}{2}\sin(2k-2)\varphi,$$

and since all terms with the exception of last vanish, we have that

$$S\sin\varphi = \frac{1}{2}\sin 2k\varphi \quad \text{therefore} \quad S = \frac{\sin 2k\varphi}{2\sin\varphi}.$$

Now reversing our substitution $\varphi = \frac{m\pi}{2k}$, we have that $2k\varphi = m\pi$, and since m is even

$$\sin 2k\varphi = \sin m\pi = 0,$$

and thus the above sum $S = 0$.

On the other hand when n is odd, we set $n = 2k+1$, and thus letting $\frac{m\pi}{2k+1} = \varphi$, it is necessary to show that:

$$\cos\varphi + \cos 3\varphi + \cdots + \cos(2k-1)\varphi + \frac{1}{2}\cos m\pi = 0.$$

Hence, by the proceeding sum, this sum becomes

$$\frac{\sin 2k\varphi}{2\sin\varphi} + \frac{1}{2}\cos m\pi = \frac{\sin 2k\varphi}{2\sin\varphi} + \frac{1}{2}\cos(2k+1)\varphi,$$

and due to the fact that

$$\sin 2k\varphi = \sin(2k+1)\varphi\cos\varphi - \cos(2k+1)\varphi\sin\varphi$$

this sum will become

$$\frac{\sin(2k+1)\varphi\cos\varphi}{2\sin\varphi}.$$

But since $(2k+1)\varphi = m\pi$, it is evident that this sum is also equal to zero.

9. Having demonstrated that the logarithmic parts of our integral

$$\int \frac{x^{m-1}dx}{1+x^n}$$

vanish when we let $x \to \infty$, it is now necessary to find the values of the remaining trigonometric parts.

Furthermore, since each term in what remains of the sum in (3) contains the factor $\tan\frac{x\sin\varphi}{1-x\cos\varphi}$, we see that (3) vanishes entirely when $x = 0$, which can be seen immediately from the left side of (3), since it corresponds to integration over a null interval. Augmenting x just until it reaches the value $x = \frac{1}{\cos\varphi}$ then results in arctan of a right angle which diverges, and if one increases x even more, the angle will become obtuse. Therefore, letting $x \to \infty$, we will have $\tan\frac{x\sin\varphi}{1-x\cos\varphi} = \arctan\frac{-\sin\varphi}{\cos\varphi} = \pi - \varphi$; and hence taken together the remaining parts of (3) become

$$\sum_{\substack{1\le i\le n\\ i \text{ odd}}} \frac{2}{n}(\pi-\varphi)\sin\frac{im\pi}{n} = \frac{2\pi}{n}\sum_{\substack{1\le i\le n\\ i \text{ odd}}}\sin\frac{im\pi}{n} - \frac{2\pi}{n^2}\sum_{\substack{1\le i\le n\\ i \text{ odd}}} i\sin\frac{im\pi}{n}.$$

Our goal is thus to find the sum of the two series on the right hand side of this equality.

10. First let $n = 2k$ be even, then setting $\frac{m\pi}{2k} = \varphi$, the first series will be:

$$\sin\varphi + \sin 3\varphi + \sin 5\varphi + \cdots + \sin(2k-1) = s,$$

that which, being mulitplied by $\sin\varphi$, gives:

$$\left.\begin{array}{l} \frac{1}{2} - \frac{1}{2}\cos 2\varphi - \frac{1}{2}\cos 4\varphi - \frac{1}{2}\cos 6\varphi - \cdots - \frac{1}{2}\cos 2k\varphi \\ \quad +\frac{1}{2}\cos 2\varphi + \frac{1}{2}\cos 4\varphi + \frac{1}{2}\cos 6\varphi \cdots\cdots\cdots\cdots\cdots \end{array}\right\} = s\sin\varphi$$

from this we derive

$$s = \frac{1-\cos 2k\varphi}{2\sin\varphi} = \frac{1-\cos m\pi}{2\sin\frac{m\pi}{2k}}.$$

And thus, having already derived:

$$\cos\varphi + \cos 3\varphi + \cos 5\varphi + \cdots + \cos(2k-1)\varphi = \frac{\sin 2k\varphi}{2\sin\varphi},$$

differentiation gives:

$$\sin\varphi + 3\sin 3\varphi + 5\sin 5\varphi + \cdots + (2k-1)\sin(2k-1)\varphi = \frac{-2k\cos 2k\varphi}{2\sin\varphi} + \frac{\sin 2k\varphi}{2\sin^2\varphi}.$$

Now setting $\varphi = \frac{m\pi}{n}$, and undoing the substitution $2k = n$, our two series will become:

$$\frac{2\pi}{n}\frac{1-\cos m\pi}{2\sin\frac{m\pi}{n}} - \frac{2\pi}{n^2}\left(\frac{-n\cos m\pi}{2\sin\frac{m\pi}{n}} + \frac{\sin m\pi}{2\sin^2\frac{m\pi}{n}}\right),$$

of which a reduction then gives:

$$\frac{\pi}{n\sin\frac{m\pi}{n}}\left(1 - \frac{\sin m\pi}{n\sin\frac{m\pi}{n}}\right) = \frac{\pi}{n\sin\frac{m\pi}{n}},$$

due to the fact that $\sin m\pi = 0$. A similar derivation shows that the same value is obtained when n is odd.

11. Now that we have rigorously demonstrated (2), then provided that $m \leq n$, we consider the substitution

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^n} = \int_0^1 \frac{z^{m-1}}{\sqrt[n]{(1-z^n)^m}}$$
$$x = \frac{z}{\sqrt[n]{1-z^n}}$$
$$dx = \frac{dz}{\sqrt[n]{(1-z^n)^{n+1}}}.$$

Hence we obtain the equality

$$\int_0^1 \frac{z^{m-1}dz}{\sqrt[n]{(1-z^n)^m}} = \frac{\pi}{n\sin\frac{m\pi}{n}}, \tag{4}$$

for $m \leq n$.

12. Upon computing a number of particular cases of (4):

$$\int_0^1 \frac{zdz}{\sqrt[3]{(1-z^3)^2}} = \frac{\pi}{3\sin\frac{2\pi}{3}} \quad \int_0^1 \frac{dz}{\sqrt[3]{1-z^3}} = \frac{\pi}{3\sin\frac{\pi}{3}} \quad \int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{2}$$
$$\int_0^1 \frac{z^2dz}{\sqrt[4]{(1-z^3)^3}} = \frac{\pi}{4\sin\frac{3\pi}{4}} \quad \int_0^1 \frac{dz}{\sqrt[4]{1-z^4}} = \frac{\pi}{4\sin\frac{\pi}{4}} \quad \int_0^1 \frac{z^2dz}{\sqrt[5]{(1-z^5)^3}} = \frac{\pi}{5\sin\frac{3\pi}{5}}$$
$$\int_0^1 \frac{zdz}{\sqrt[5]{(1-z^5)^2}} = \frac{\pi}{5\sin\frac{2\pi}{5}} \quad \int_0^1 \frac{dz}{\sqrt[5]{1-z^5}} = \frac{\pi}{5\sin\frac{\pi}{5}} \quad \int_0^1 \frac{z^3dz}{\sqrt[5]{(1-z^5)^4}} = \frac{\pi}{5\sin\frac{4\pi}{5}}$$
$$\int_0^1 \frac{z^4dz}{\sqrt[6]{(1-z^6)^5}} = \frac{\pi}{6\sin\frac{5\pi}{6}} \quad \int_0^1 \frac{dz}{\sqrt[6]{1-z^6}} = \frac{\pi}{6\sin\frac{\pi}{6}},$$

we observe that these integrals are quite remarkable, however we still lack expedient methods for their derivation.

13. Given the equality in (4), we search for a series expansion of the left hand side, which, due to the fact that

$$(1-z^n)^{\frac{-m}{n}} = \sum_{i=0}^{\infty}(-1)^i\binom{-\frac{m}{n}}{i}z^{in},$$

for $\binom{\alpha}{i}$ the generalized binomial coefficients, we are able to derive as follows:

$$\begin{aligned}\int_0^1 \frac{z^{m-1}dz}{\sqrt[n]{(1-z^n)^m}} = \int_0^1 \sum_{i=0}^{\infty}(-1)^i\binom{-\frac{m}{n}}{i}z^{in+m-1}dz &= \sum_{i=0}^{\infty}(-1)^i\binom{-\frac{m}{n}}{i}\int_0^1 z^{in+m-1}dz \\ &= \sum_{i=0}^{\infty}(-1)^i\binom{-\frac{m}{n}}{i}\frac{1}{in+m}.\end{aligned}$$

We therefore obtain the following correspondence:

$$\int_0^1 \frac{z^{m-1}dz}{\sqrt[n]{(1-z^n)^m}} = \frac{\pi}{n\sin\frac{m\pi}{n}} = \sum_{i=0}^{\infty}(-1)^i\binom{-\frac{m}{n}}{i}\frac{1}{in+m}.$$

Moreover, we will see in *Section 14* that this same integral can also be expressed by the following infinite product:

$$\int_0^1 \frac{z^{m-1}dz}{\sqrt[n]{(1-z^n)^m}} = \frac{\pi}{n\sin\frac{m\pi}{n}} = \frac{1}{n-m}\prod_{i=1}^{\infty}\frac{i^2n^2}{((i-1)n+m)((i+1)n-m)} \qquad (5)$$

In the special case where $m=1$ and $n=2$, we have Wallis' product for π:

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\frac{10\cdot 10}{9\cdot 11}\cdots.$$

Next, setting $m=1$ and $n=6$, we have

$$\frac{\pi}{3} = \frac{1}{5}\cdot\frac{6^2}{1\cdot 14}\cdot\frac{12^2}{7\cdot 17}\cdot\frac{18^2}{13\cdot 23}\cdot\frac{24^2}{19\cdot 29}\cdots,$$

or for π itself,

$$\pi = \frac{18}{5}\cdot\frac{6\cdot 12}{7\cdot 11}\cdot\frac{12\cdot 18}{13\cdot 17}\cdot\frac{18\cdot 24}{19\cdot 23}\cdot\frac{24\cdot 30}{25\cdot 29}\cdots.$$

14. The previous section's products, being the very as those I found in my introduction (different paper), provide an alternative route to discovering the family of integrals in (1). As such, we have the following product expansions for sine and cosine:

$$\begin{aligned}\sin\frac{m\pi}{n} &= \frac{m\pi}{n}\prod_{i=1}^{\infty}\left(1-\frac{m^2}{i^2n^2}\right) = \frac{m\pi}{n}\prod_{i=1}^{\infty}\frac{(in-m)(in+m)}{i^2n^2}, \\ \cos\frac{m\pi}{n} &= \prod_{i=1}^{\infty}\left(1-\frac{4m^2}{(2i-1)^2n^2}\right) = \prod_{i=1}^{\infty}\frac{((2i-1)n-2m)((2i-1)n+2m)}{(2i-1)^2n^2}\Big),\end{aligned}$$

formulas of which the former leads to

$$\frac{\pi}{n \sin \frac{m\pi}{n}} = \frac{1}{m} \prod_{i=1}^{\infty} \frac{i^2 n^2}{(in-m)(in+m)}, \tag{6}$$

and which, upon replacing m with $n-m$, we can immediately see is equal to (5).

We would have thus also arrived at the very same integral in (5), if we had instead begun with the product expansion in (6) and searched for an integral whose value it equaled. However, given that I have already provided methods of evaluation for the above integrals, and in certain cases by forming product expansions, it is not necessary at this time to reverse this method and pass from product expansions back to integrals.

15. Following the demonstration in (5), if we make the substitutions $\alpha = \mu - v = m$, and $\mu = n$, then we will have:

$$\int_0^1 x^{\alpha-1}(1-x^{\mu})^{\frac{v-\mu}{\mu}}\,dx = \frac{1}{v} \cdot \frac{\mu(\alpha+v)}{\alpha(\mu+v)} \cdot \frac{2\mu(\alpha+v+\mu)}{(\alpha+\mu)(2\mu+v)} \cdot \frac{3\mu(\alpha+v+2\mu)}{(\alpha+2\mu)(3\mu+v)} \cdot \cdots ,$$

and thus considering the quotient of two such integrals, we obtain the integral ratio:

$$\frac{\int_0^1 x^{\alpha-1}(1-x^{\mu})^{\frac{v-\mu}{\mu}}\,dx}{\int_0^1 x^{\beta-1}(1-x^{\mu})^{\frac{v-\mu}{\mu}}\,dx} = \frac{\beta(\alpha+v)}{\alpha(\beta+v)} \cdot \frac{(\beta+\mu)(\alpha+v+\mu)}{(\alpha+\mu)(\beta+v+\mu)} \cdot \frac{(\beta+2\mu)(\alpha+v+2\mu)}{(\alpha+2\mu)(\beta+v+2\mu)} \cdot \cdots ,$$

and then, more generally:

$$\frac{\int_0^1 x^{\alpha-1}(1-x^{\mu})^{\frac{v-\mu}{\mu}}\,dx}{\int_0^1 x^{\beta-1}(1-x^{\mu})^{\frac{\lambda-\mu}{\mu}}\,dx} = \frac{\lambda}{v} \cdot \frac{\beta(\alpha+v)(\lambda+\mu)}{\alpha(\beta+\lambda)(v+\mu)} \cdot \frac{(\beta+\mu)(\alpha+v+\mu)(\lambda+2\mu)}{(\alpha+\mu)(\beta+\lambda+\mu)(v+2)} \cdot \frac{(\beta+2\mu)(\alpha+v+2\mu)(\lambda+3\mu)}{(\alpha+2\mu)(\beta+\lambda+2\mu)(v+3\mu)} \cdot \cdots .$$

Therefore, given such a product, we could of course work backwards towards an integral, or the quotient of two.

16. Now let us modify (5) slightly in order to compare the infinite product

$$\frac{(n-m)\pi}{n \sin \frac{m\pi}{n}} = \prod_{i=1}^{\infty} \frac{i^2 n^2}{((i-1)n+m)((i+1)n-m)},$$

with the infinite product of the integral:

$$v \int_0^1 x^{\alpha-1}(1-x^{\mu})^{\frac{v-\mu}{\mu}}\,dx = \prod_{i=1}^{\infty} \frac{i\mu(\alpha+v+(i-1)\mu)}{(\alpha+(i-1)\mu)(i\mu+v)},$$

Since the growth of the former is $O(n)$, while the latter is $O(\mu)$, we will begin by setting $\mu = n$. Therefore if we require that the integral take the form given in (5), we must have $\alpha + v = n$, and then we will either set $\alpha = m$, implying that $\mu + v = 2n - m$, or $\mu + v = m$ which will imply that $\alpha = 2n - m$. Thus the first case can be summarized as $\alpha = m$, $\mu = n$, and $v = n - m$, and the latter case summarized as $\alpha = 2n - m$, $\mu = n$, and $v = m - n$. Therefore we will have

$$\begin{aligned} \text{either} \quad \frac{(n-m)\pi}{n \sin \frac{m\pi}{n}} &= (n-m) \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}, \\ \text{or} \quad \frac{(n-m)\pi}{n \sin \frac{m\pi}{n}} &= (m-n) \int \frac{x^{2n-m-1}dx}{(1-x^n)^{\frac{2n-m}{n}}}, \end{aligned}$$

where the latter integral will not converge whenever $n > m$.

17. Provided that $m \leq n$, we have found two distinct routes for the rigorous demonstration of (4):

$$\int_0^1 \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^m}} = \frac{\pi}{n \sin \frac{m\pi}{n}}.$$

First, by applying integration by substitution to the integral given in (1):

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^n},$$

whose value is derived from its series representation given at (3). Second, by writing this integral as the infinite product in (5), and then finding its limit.

Now in (5) if we replace m with $n - m$, we will have

$$\int \frac{z^{n-m-1}dz}{(1-z^n)^{\frac{n-m}{n}}} = \frac{1}{m} \prod_{i=1}^{\infty} \frac{i^2 n^2}{i^2 n^2 - m^2},$$

and by consequence, given that $\sin \frac{(n-m)\pi}{n} = \sin \frac{m\pi}{n}$, we obtain

$$\int_0^1 \frac{z^{m-1}dz}{(1-z^n)^{\frac{m}{n}}} = \int_0^1 \frac{z^{n-m-1}dz}{(1-z^n)^{\frac{n-m}{n}}} = \frac{\pi}{n \sin \frac{m\pi}{n}}.$$

Then reversing the substitution done in *Section 11*, we will have:

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^n} = \int_0^\infty \frac{x^{n-m-1}dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}.$$

18. We see also how this same infinite product:

$$\prod_{i=1}^{\infty} \frac{i^2 n^2}{i^2 n^2 - m^2} = \frac{m\pi}{n \sin \frac{m\pi}{n}},$$

can be expressed as the quotient of two integrals. To this effect, it's necessary to set $\mu = n$ and $\frac{\beta(\alpha+v)}{\alpha(\beta+v)} = \frac{n^2}{n^2-m^2}$, therefore $\beta = n$; $\alpha + v = n$; $\alpha = n - m$ and $\beta + v = n + m$; from here we let $\alpha = n - m$; $\beta = n$; $v = m$ and $\mu = n$ and thus we have:

$$\frac{m\pi}{n \sin \frac{m\pi}{n}} = \frac{\int_0^1 x^{n-m-1}(1-x^n)^{\frac{m-n}{n}}\, dx}{\int_0^1 x^{n-1}(1-x^n)^{\frac{m-n}{n}}\, dx},$$

where the denominator integrates to $\frac{1}{m}$. The more general integral does not lead to other integrals; however there are other methods of making these integrals more general.

19. Multiplying two integrals in general and in the case where, the value of this product

$$v\mathfrak{n} \int_0^1 x^{\alpha-1}(1-x^n)^{\frac{v-n}{n}}\, dx \cdot \int_0^1 x^{\mathfrak{n}-1}(1-x^n)^{\frac{\mathfrak{n}-n}{n}}\, dx$$

will be

$$\frac{n^2(\alpha+v)(\alpha+\mathfrak{n})}{\alpha\mathfrak{n}(v+n)(\mathfrak{n}+n)} \cdot \frac{4n^2(\alpha+v+n)(\alpha+\mathfrak{n}+n)}{(\alpha+n)(\mathfrak{a}+n)(v+2n)(\mathfrak{n}+2n)} \cdot \cdots$$

that which is noted to be equal to the following:

$$\frac{n^2}{(n-m)(n+m)} \cdot \frac{4n^2}{(2n-m)(2n+m)} \cdot \cdots = \frac{m\pi}{n \sin \frac{m\pi}{n}}.$$

Let, for this purpose, $\alpha = n - m$; $\alpha = n + m$; and letting also the following:

$$\alpha + v = v + n - m = \mathfrak{n} + n; \quad \mathfrak{a} + \mathfrak{n} = \mathfrak{n} + n + m = v + n;$$

from which we derive $v - n = m$. Thus let $v = k + \frac{1}{2}m$ and $\mathfrak{n} = k - \frac{1}{2}m$; and we will have, by taking k an arbitrary number:

$$\begin{aligned}(k^2 - \frac{1}{4}m^2) \int_0^1 x^{n-m-1}(1-x^n)^{\frac{2k+m-2n}{2n}}\, dx \cdot \int_0^1 x^{n+m-1}(1-x^n)^{\frac{2k-m-2n}{2n}}\, dx \\ = \frac{m\pi}{n \sin \frac{m\pi}{n}}.\end{aligned} \tag{7}$$

20. Therefore in (7) we have the product of two integrals which are equal to $\frac{m\pi}{n \sin \frac{m\pi}{n}}$; and consequently assuming we take k in such a way that these integrals are finite, then they will reduce to the expression $\frac{m\pi}{n \sin \frac{m\pi}{n}}$. For example, setting $2k = m + 2n$, taking into account the fact that $\int_0^1 x^{n+m-1}dx = \frac{1}{n+m}$, and $k^2 - \frac{1}{4}m^2 = n(n+m)$ we will have:

$$n \int_0^1 x^{n-m-1}(1-x^n)^{\frac{m}{n}}\, dx = \frac{m\pi}{n \sin \frac{m\pi}{n}}.$$

Another example, consider setting $2k = m + 4n$, then

$$\int_0^1 x^{n+m-1}(1-x^n)dx = \frac{n}{(n+m)(2n+m)},$$

and $k^2 - \frac{1}{4}m^2 = 2n(m+2n)$, we will have:

$$\frac{2n^2}{n+m}\int_0^1 x^{n-m-1}(1-x^n)^{\frac{m+n}{n}}dx = \frac{m\pi}{n\sin\frac{m\pi}{n}}.$$

Therefore, also putting $n-m$ in place of m, we will have the following two forms:

$$\begin{aligned}
\frac{\pi}{n\sin\frac{m\pi}{n}} &= \frac{n}{m}\int_0^1 x^{n-m-1}(1-x^n)^{\frac{m}{n}}dx = \frac{n}{n-m}\int_0^1 x^{m-1}(1-x^n)^{\frac{n-m}{n}}dx \\
\frac{\pi}{n\sin\frac{m\pi}{n}} &= \frac{2n^2}{m(n+m)}\int_0^1 x^{n-m-1}(1-x^n)^{\frac{m+n}{n}}dx \\
&= \frac{2n^2}{(n-m)(2n-m)}\int_0^1 x^{m-1}(1-x^n)^{\frac{2n-m}{n}}dx.
\end{aligned}$$

21. Now, since

$$\int_0^1 x^{n+m-1}(1-x^n)^{\frac{2k-m-2n}{2n}}dx = \frac{2m}{2k+m}\int x^{m-1}(1-x^n)^{\frac{2k-m-2n}{2n}}dx,$$

if we substitute this value, we will have:

$$(k-\frac{1}{2}m)\int x^{n-m-1}(1-x^n)^{\frac{2k+m-2n}{2n}}dx\cdot\int_0^1 x^{m-1}(1-x^n)^{\frac{2k-m-2n}{2n}}dx = \frac{\pi}{n\sin\frac{m\pi}{n}},$$

and this value remains the same, although we write $n-m$ in place of m. Let $m = 1$ and $n = 2$ and we will have:

$$(k-\frac{1}{2})\int_0^1 (1-x^2)^{\frac{2k-3}{4}}dx\cdot\int(1-x^2)^{\frac{2k-5}{4}}dx = \frac{\pi}{2},$$

where it is remarkable that this equality occurs, no matter what number we set k equal to. As an example, let $k = 1$, or $k = 2$ and one will have:

$$\begin{aligned}
\frac{1}{2}\int_0^1 \frac{dx}{\sqrt[4]{1-x^2}}\cdot\int_0^1 \frac{dx}{\sqrt[4]{(1-x^2)^3}} &= \frac{\pi}{2}, \\
\frac{3}{2}\int_0^1 \sqrt[4]{1-x^2}dx\cdot\int_0^1 \frac{dx}{\sqrt[4]{1-x^2}} &= \frac{\pi}{2},
\end{aligned}$$

and then setting $k = \frac{1}{2} + \sqrt{2}$

$$\int_0^1 (1-x^2)^{\frac{\sqrt{2}-1}{2}}dx\cdot\int_0^1 (1-x^2)^{\frac{\sqrt{2}-2}{2}}dx = \frac{\pi}{2\sqrt{2}}.$$

Taking note of the irrationality of the exponents, this equality is remarkable.

22. One can still transform in several ways the formulas that we have found, since, setting $1 - x^{2n} = y^{2n}$, in such a way that $x = \sqrt[n]{1-y^{2n}}$ and $dx = 2y^{2n-1}(1-y^{2n})dy$ the terms of the integral, which were beforehand set equal to $x = 0$ and $x = 1$, are at present reversed, to know that $y = 1$ and $y = 0$ which becomes the same. From here we conclude

$$(4k-2m)\int_0^1 y^{2k+m-1}(1-y^{2n})^{\frac{-m}{n}}\,dy \cdot \int_0^1 y^{2k-m-1}(1-y^{2n})^{\frac{m-n}{n}}\,dy = \frac{\pi}{n\sin\frac{m\pi}{n}},$$

when we will have set $y = 1$ after integration; or in other words

$$(4k^2-m^2)\int_0^1 y^{2k-m-1}(1-y^{2n})^{\frac{-m}{n}} \cdot \int_0^1 y^{2k-m-1}(1-y^{2n})^{\frac{m}{n}}\,dy = \frac{\pi}{n\sin\frac{m\pi}{n}},$$

by the reduction of these integrals. Therefore if $m = 1$ and $n = 2$, we will have:

$$(4k-2)\int_0^1 \frac{y^{2k}dy}{\sqrt{1-y^4}} \cdot \int_0^1 \frac{y^{2k-2}dy}{\sqrt{1-y^4}} = \frac{\pi}{2},$$

and consequently, if $k = 1$,

$$\int_0^1 \frac{y^2dy}{\sqrt{1-y^4}} \cdot \int_0^1 \frac{dy}{\sqrt{1-y^4}} = \frac{\pi}{4}.$$

23. Now, since the angle $\frac{m\pi}{n}$ depends only on the numbers m and n, we will have $\sin\frac{m\pi}{n} = 1$, if $m = \frac{1}{2}n$, as long as we don't need to determine n. Let therefore $m = \frac{1}{2}n$, and in order to avoid the fractions, $2k = m + \lambda$; from where we will derive the following theorem:

$$\int_0^1 \frac{y^{\lambda+n-1}dy}{\sqrt{1-y^{2n}}} \cdot \int_0^1 \frac{y^{\lambda-1}dy}{\sqrt{1-y^{2n}}} = \frac{\pi}{2\lambda n},$$

$$\int_0^1 \frac{y^{\lambda+n-1}dy}{\sqrt{1-y^{2n}}} \cdot \int_0^1 y^{\lambda-1}(1-y^{2n})dy = \frac{\pi}{2\lambda(\lambda+n)}.$$

In the same vein, setting more generally $2k = \lambda + m$, we will have:

$$\int_0^1 y^{\lambda+2m-1}(1-y^{2n})^{\frac{-m}{n}}\,dy \cdot \int_0^1 y^{\lambda-1}(1-y^{2n})^{\frac{m-n}{n}}\,dy = \frac{\pi}{2\lambda n\sin\frac{m\pi}{n}}, \quad \text{or}$$

$$\int_0^1 y^{\lambda+2m-1}(1-y^{2n})^{\frac{-m}{n}}\,dy \cdot \int_0^1 y^{\lambda-1}(1-y^{2n})^{\frac{m}{n}}\,dy = \frac{m\pi}{\lambda n(\lambda+2m)\sin\frac{m\pi}{n}},$$

where the number λ is arbitrary, in such a way that one can even give it an irrational value. Let $m = \mu k$ and $n = vk$, and one will have:

$$\int_0^1 y^{\lambda+2m\mu k-1}(1-y^{2vk})^{\frac{-\mu}{v}}\,dy \cdot \int_0^1 y^{\lambda-1}(1-y^{2vk})^{\frac{\mu-v}{v}}\,dy = \frac{\pi}{2\lambda vk\sin\frac{\mu\pi}{v}}, \quad \text{or}$$

$$\int_0^1 y^{\lambda+2m\mu k-1}(1-y^{2vk})^{\frac{-\mu}{v}}\,dy \cdot \int_0^1 y^{\lambda-1}(1-y^{2vk})^{\frac{\mu}{v}}\,dy = \frac{\mu\pi}{\lambda v(\lambda+2\mu k)\sin\frac{\mu\pi}{v}}.$$

24. Moreover, setting $2k=\alpha$ in order to have this equality

$$\int_0^1 y^{\lambda+\mu\alpha-1}(1-y^{v\alpha})^{\frac{-\mu}{v}}dy\cdot\int_0^1 y^{\lambda-1}(1-y^{v\alpha})^{\frac{\mu-v}{v}}dy=\frac{\pi}{\lambda v\alpha\sin\frac{\mu\pi}{v}},$$

from this we have the following principal cases:

$$\begin{aligned}
&\int_0^1\frac{y^{\lambda-\alpha-1}dy}{\sqrt{1-y^{2\alpha}}}\cdot\int_0^1\frac{y^{\lambda-1}dy}{\sqrt{1-y^{2\alpha}}}=\frac{\pi}{2\lambda\alpha},\\
&\int_0^1\frac{y^{\lambda-\alpha-1}dy}{\sqrt[3]{1-y^{3\alpha}}}\cdot\int_0^1\frac{y^{\lambda-1}dy}{\sqrt[3]{(1-y^{3\alpha})^2}}=\frac{2\pi}{3\lambda\alpha\sqrt{3}},\\
&\int_0^1\frac{y^{\lambda-\alpha-1}dy}{\sqrt[4]{1-y^{4\alpha}}}\cdot\int_0^1\frac{y^{\lambda-1}dy}{\sqrt[4]{(1-y^{3\alpha})^3}}=\frac{\pi}{2\lambda\alpha\sqrt{2}},\\
&\int_0^1\frac{y^{\lambda-2\alpha-1}dy}{\sqrt[3]{(1-y^{3\alpha})^2}}\cdot\int_0^1\frac{y^{\lambda-1}dy}{\sqrt[3]{1-y^{3\alpha}}}=\frac{2\pi}{3\lambda\alpha\sqrt{3}},\\
&\int_0^1\frac{y^{\lambda-3\alpha-1}dy}{\sqrt[4]{(1-y^{4\alpha})^3}}\cdot\int_0^1\frac{y^{\lambda-1}dy}{\sqrt[4]{1-y^{4\alpha}}}=\frac{\pi}{2\lambda\alpha\sqrt{2}}.
\end{aligned}$$

25 Just as the infinite product expansion for sine has led us to the above integrals, we treat in the same way the infinite product expansion found for cosine, which can be written in the following form:

$$\cos\frac{m\pi}{n}=\prod_{i=1}^{\infty}\frac{((2i-1)n-2m)((2i-1)n+2m)}{(2i-1)^2n^2},$$

where since the sequence of numbers in both the numerator and denominator of this product are only odd numbers, we will not be able to express it with a single integral. We therefore search for two integrals whose quotient expresses this value, and one sees first that it's necessary to set $\mu=2n$. Thus let $\frac{\beta(\alpha+v)}{\alpha(\beta+v)}=\frac{(n-2m)(n+2m)}{n^2}$, and we will have $\alpha=n$; $\beta=n-v$ and $v=2m$; in such a way that $\beta=n-2m$. As a consequence we will have:

$$\frac{\int_0^1 x^{n-1}(1-x^{2n})^{\frac{m-n}{n}}dx}{\int_0^1 x^{n-2m-1}(1-x^{2n})^{\frac{m-n}{n}}dx}=\cos\frac{m\pi}{n}=\sin\frac{(n-2m)\pi}{2n}.$$

Therefore setting $m=\lambda\mu$ and $n=\lambda v$, we will have

$$\frac{\int_0^1 x^{\lambda v-1}(1-x^{2\lambda v})^{\frac{\mu-v}{v}}dx}{\int_0^1 x^{\lambda v-2\lambda\mu-1}(1-x^{2\lambda v})^{\frac{\mu-v}{v}}dx}=\cos\frac{m\pi}{v}=\sin\frac{(v-2\mu)\pi}{2v}.$$

26. We consider the most simple cases:

I. If $m=1,n=2:\frac{\int_0^1 x(1-x^4)^{-\frac{1}{2}}dx}{\int_0^1 x^{-1}(1-x^4)^{\frac{-1}{2}}dx}=\cos\frac{\pi}{2}=0.$

II. If $m = 1, n = 3$: $\frac{\int_0^1 x^2(1-x^6)^{-\frac{2}{3}}\,dx}{\int_0^1 (1-x^6)^{\frac{-2}{3}}\,dx} = \cos\frac{\pi}{3} = \frac{1}{2}$.

III. If $m = \frac{1}{2}, n = 2$: $\frac{\int_0^1 x(1-x^4)^{-\frac{-3}{4}}\,dx}{\int_0^1 (1-x^4)^{\frac{-3}{4}}\,dx} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

IV. If $m = \frac{1}{2}, n = 3$: $\frac{\int_0^1 x(1-x^6)^{-\frac{5}{6}}\,dx}{\int_0^1 (1-x^6)^{\frac{-5}{6}}\,dx} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$.

After performing the appropriate variable substitutions, from the second we can derive the equality:

$$\int_0^1 \frac{dx}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{2}\int_0^1 \frac{dx}{\sqrt[3]{(1-x^6)^2}},$$

the third reduces to

$$\int_0^1 \frac{dx}{\sqrt[4]{(1-x^2)^2}} = \sqrt{2}\int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}},$$

and the fourth to

$$\int_0^1 \frac{dx}{\sqrt[6]{(1-x^2)^5}} = \frac{3\sqrt{3}}{4}\int_0^1 \frac{dx}{\sqrt[6]{(1-x^3)^5}}.$$

27. We can also make the substitutions to find a slightly simpler form for the left and right hand side of the second:

$$\int_0^1 \frac{dx}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{2}\int_0^1 \frac{dx}{\sqrt{1-x^3}}, \qquad \text{upon setting } x^3 \text{ in place of } 1-x^2,$$
$$\int_0^1 \frac{dx}{\sqrt[3]{(1-x^6)^2}} = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt[6]{(1-x^3)^6}}, \qquad \text{upon setting } x^3 \text{ in place of } 1-x^6,$$

and from here we will have the following equalities:

$$\int_0^1 \frac{dx}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{2}\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{3}{2}\int_0^1 \frac{dx}{\sqrt[3]{(1-x^6)^2}} = \frac{3}{4}\int_0^1 \frac{dx}{\sqrt[6]{(1-x^3)^5}},$$
$$\int_0^1 \frac{dx}{\sqrt[4]{(1-x^2)^3}} = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \sqrt{2}\int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}},$$
$$\int_0^1 \frac{dx}{\sqrt[6]{(1-x^2)^5}} = 3\int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{3\sqrt{3}}{4}\int_0^1 \frac{dx}{\sqrt[6]{(1-x^3)^5}} = \frac{3\sqrt{3}}{2}\int_0^1 \frac{dx}{\sqrt[3]{(1-x^6)^2}}.$$

28. Via the same transformation, we find in general:

$$\int_0^1 \frac{x^{n-1}dx}{(1-x^{2n})^{\frac{n-m}{m}}} = \frac{1}{2}\int_0^1 \frac{x^{m-1}dx}{(1-x^n)^{\frac{1}{2}}} = \cos\frac{m\pi}{n}\cdot\int_0^1 \frac{x^{n-2m-1}dx}{(1-x^{2n})^{\frac{n-m}{n}}}$$
$$= \frac{1}{2}\cos\frac{m\pi}{n}\cdot\int_0^1 \frac{x^{m-1}dx}{(1-x^n)^{\frac{n+2m}{2n}}};$$

and in particular we obtain another version of (4):

$$\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^n}} = \cos\frac{m\pi}{n}\int_0^1 \frac{x^{m-1}dx}{\sqrt[2n]{(1-x^n)^{n+2m}}}. \tag{8}$$

Therefore, since the left side of (8) is the simplest version of those considered at the beginning of this section, since its denominator contains only a second root, we will have the following reductions:

$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[n]{(1-x^{2n})^{n-m}}} = \frac{1}{2}\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^n}},$$
$$\int_0^1 \frac{x^{n-2m-1}dx}{\sqrt[n]{(1-x^{2n})^{n-m}}} = \frac{1}{2\cos\frac{m\pi}{n}}\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^n}},$$
$$\int_0^1 \frac{x^{m-1}dx}{\sqrt[2n]{(1-x^n)^{n+2m}}} = \frac{1}{\cos\frac{m\pi}{n}}\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^n}},$$

of which the first immediately self-evident given the substitution $u = x^2$ and the equality $n = 2m$, on the other hand the two follow similar derivations to that given above.

29. Recalling from *Section 14* the product expansion for sine, and taking the appropriate quotient, we obtain:

$$\frac{\sin\frac{m\pi}{2n}}{\sin\frac{k\pi}{2n}} = \prod_{i=0}^{\infty}\frac{(2in+m)(2(i+1)n-m)}{(2in+k)(2(i+1)n-k)},$$

which we will show can be reduced to a ratio of of two integrals. First however we must set $\mu = 2n$ and set

$$\frac{\beta(\alpha+v)}{\alpha(\beta+v)} = \frac{m(2n-m)}{k(2n-k)},$$

which can be done in four ways:

I. $\alpha = k$; $\beta = m$; $v = 2n-m-k$; $\frac{v-\mu}{\mu} = \frac{-m-k}{2n}$.

II. $\alpha = k$; $\beta = 2n-m$; $v = m-k$; $\frac{v-\mu}{\mu} = \frac{m-k-2n}{2n}$.

III. $\alpha = 2n - k;\ \ \beta = m;\ \ v = k - m;\ \ \frac{v-\mu}{\mu} = \frac{k-m-2n}{2n}.$

IV. $\alpha = 2n - k;\ \ \beta = 2n - m;\ \ v = m + k - 2n;\ \ \frac{v-\mu}{\mu} = \frac{m+k-4n}{2n}.$

And thus from *Section 15* we can conclude:

$$\frac{\int_0^1 x^{\alpha-1}(1-x^\mu)^{\frac{v-\mu}{\mu}}\,dx}{\int_0^1 x^{\beta-1}(1-x^\mu)^{\frac{v-\mu}{\mu}}\,dx} = \frac{\sin\frac{m\pi}{2n}}{\sin\frac{k\pi}{2n}},$$

and then by the more general form given in Section 15:

$$\frac{\int_0^1 x^{v-1}(1-x^\mu)^{\frac{\alpha-\mu}{\mu}}\,dx}{\int_0^1 x^{v-1}(1-x^\mu)^{\frac{\beta-\mu}{\mu}}\,dx} = \frac{\sin\frac{m\pi}{2n}}{\sin\frac{k\pi}{2n}}. \tag{9}$$

30. Equation (9) furnishes the following reformulations:

$$\begin{aligned}
\sin\frac{k\pi}{2n}\cdot\int_0^1 \frac{x^{2n-m-k-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-k}}} &= \sin\frac{m\pi}{2n}\cdot\int_0^1 \frac{x^{2n-m-k-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}},\\
\sin\frac{k\pi}{2n}\cdot\int_0^1 \frac{x^{m-k-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-k}}} &= \sin\frac{m\pi}{2n}\cdot\int_0^1 \frac{x^{m-k-1}dx}{\sqrt[2n]{(1-x^{2n})^{m}}},\\
\sin\frac{k\pi}{2n}\cdot\int_0^1 \frac{x^{k-m-1}dx}{\sqrt[2n]{(1-x^{2n})^{k}}} &= \sin\frac{m\pi}{2n}\cdot\int_0^1 \frac{x^{k-m-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}},\\
\sin\frac{k\pi}{2n}\cdot\int_0^1 \frac{x^{m+k-2n-1}dx}{\sqrt[2n]{(1-x^{2n})^{k}}} &= \sin\frac{m\pi}{2n}\cdot\int_0^1 \frac{x^{m+k-2n-1}dx}{\sqrt[2n]{(1-x^{2n})^{m}}}.
\end{aligned}$$

In addition I will provide several examples. In fact one can derive from them above several quite remarkable reductions. Setting $k = n - m$, we have:

$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{n+m}}} = \tan\frac{m\pi}{2n}\cdot\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}}.$$

31. Next we consider the product expansion I have found for tangent:

$$\tan\frac{m\pi}{2n} = \frac{m\pi}{2(n-m)}\cdot\frac{(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdots.$$

so that we have

$$\frac{m\pi}{2(n-m)\tan\frac{m\pi}{2n}} = \frac{4n^2(n+m)(3n-m)}{3n^2(2n-m)(2n+m)}\cdot\frac{4n^2(3n+m)(5n-m)}{15n^2(4n-m)(4n+m)}\cdots.$$

which may be reduced to the product of two integrals, given in general by the formula

$$\begin{aligned}
&vn\int_0^1 x^{\alpha-1}(1-x^\mu)^{\frac{v-\mu}{\mu}}\,dx\cdot\int_0^1 x^{\alpha-1}(1-x^m)^{\frac{n-m}{m}}\,dx\\
&= \frac{\mu m(\alpha+v)(a+n)}{\alpha a(\mu+v)(m+n)}\cdot\frac{4\mu m(\alpha+v+\mu)(a+n+m)}{(\alpha+\mu)(a+m)(2\mu+v)(2m+n)}\cdots,
\end{aligned}$$

where we set both $\mu = m = 2n$, and:

$$\frac{(n+m)(3n-m)}{3n^2(2n-m)(2n+m)} = \frac{(\alpha+v)(a+n)}{\alpha a(\mu+v)(m+n)}.$$

Where one is thus required to have $\alpha + v = n + m$ and $a + n = 3n - m$, with following collections of relations which follow:

I. $v = m$; $n = -m$; $\alpha = n$; $a = 3n$; $\mu = 2n$; $m = 2n$.

II. $v = m$; $n = n$; $\alpha = n$; $a = 2n - m$, $\mu = 2n$; $m = 2n$.

III. $v = -n$; $n = -m$; $\alpha = 2n + m$; $a = 3n$; $\mu = 2n$; $m = 2n$.

IV. $v = -n$; $n = n$; $\alpha = 2n + m$; $a = 2n - m$; $\mu = 2n$; $m = 2n$.

32. Thus we have the reductions:

$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}} \cdot \int_0^1 \frac{x^{3n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n+m}}} = \frac{\pi}{2m(m-n)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}} \cdot \int_0^1 \frac{x^{2n-m-1}dx}{\sqrt[2n]{(1-x^{2n})^{n}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{2n+m-1}dx}{\sqrt[2n]{(1-x^{2n})^{3n}}} \cdot \int_0^1 \frac{x^{3n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n+m}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{2n+m-1}dx}{\sqrt[2n]{(1-x^{2n})^{3n}}} \cdot \int_0^1 \frac{x^{2n-m-1}dx}{\sqrt[2n]{(1-x^{2n})^{n}}} = \frac{m\pi}{2n^2(m-n)} \cot \frac{m\pi}{2n},$$

where we make use of the equalities:

$$\int_0^1 \frac{x^{3n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n+m}}} = \frac{-n}{m} \int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{m}}},$$
$$\int_0^1 \frac{x^{2n+m-1}dx}{\sqrt[2n]{(1-x^{2n})^{3n}}} = \frac{-m}{n} \int_0^1 \frac{x^{m-1}dx}{\sqrt[2n]{1-x^{2n}}}.$$

33. These substitutions provide us with the following formulas:

$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}} \cdot \int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{m}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{2n-m}}} \cdot \int_0^1 \frac{x^{2n+m-1}dx}{\sqrt[2n]{1-x^{2n}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{m-1}dx}{\sqrt[2n]{1-x^{2n}}} \cdot \int_0^1 \frac{x^{n-1}dx}{\sqrt[2n]{(1-x^{2n})^{m}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{m-1}dx}{\sqrt[2n]{1-x^{2n}}} \cdot \int_0^1 \frac{x^{2n-m-1}dx}{\sqrt[2n]{1-x^{2n}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n},$$

which reduce to the following simpler formulas:

$$\int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^{2n-m}}} \cdot \int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^m}} = \frac{n\pi}{2(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^{2n-m}}} \cdot \int_0^1 \frac{x^{2n-m-1}dx}{\sqrt[2n]{1-x^{2n}}} = \frac{\pi}{2(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^2}} \cdot \int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^m}} = \frac{\pi}{2(n-m)} \cot \frac{m\pi}{2n},$$
$$\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^2}} \cdot \int_0^1 \frac{x^{2n-m-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2n(n-m)} \cot \frac{m\pi}{2n}.$$

34. And furthermore by the subsequent substitutions, one finds

$$\int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^{2n-m}}} = n\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^{2n}}},$$
$$\int_0^1 \frac{dx}{\sqrt[2n]{(1-x^2)^m}} = n\int_0^1 \frac{x^{2n-m-1}dx}{\sqrt{1-x^{2n}}}.$$

Whereby all the formulas reduce to the last which is the most simple since it is only contained in the square root sign, that which, if we set $m = n - k$, it changes into this quite remarkable formula

$$\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk} \tan \frac{k\pi}{2n},$$

From here, if $k = 0$, due to the fact that $\tan \frac{k\pi}{2n} = \frac{k\pi}{2n}$, one obtains:

$$\int_0^1 \frac{x^{n-1}dx}{\sqrt{1-x^{2n}}} \cdot \int \frac{x^{n-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi^2}{4n^2}.$$

35. We consider several particular cases:

I. If $n = 1$, $k = 0$: $\int \frac{dx}{\sqrt{1-x^2}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi^2}{4}$.

II. If $n = \frac{3}{2}$, $k = \frac{1}{2}$: $\int_0^1 \frac{xdx}{\sqrt{1-x^3}} \cdot \int \frac{dx}{\sqrt{1-x^3}} = \frac{2\pi}{3} \tan \frac{\pi}{6} = \frac{2\pi}{3\sqrt{3}}$.

III. If $n = 2$, $k = 1$: $\int \frac{x^2dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4} \tan \frac{\pi}{4} = \frac{\pi}{4}$.

IV. If $n = \frac{5}{2}$, $k = \frac{1}{2}$: $\int_0^1 \frac{x^2dx}{\sqrt{1-x^5}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{2\pi}{5} \tan \frac{\pi}{10}$.

V. If $n = \frac{5}{2}$, $k = \frac{3}{2}$: $\int_0^1 \frac{x^3dx}{\sqrt{1-x^5}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^5}} = \frac{2\pi}{15} \tan \frac{3\pi}{10}$.

VI. If $n = 3$, $k = 1$: $\int_0^1 \frac{x^3 dx}{\sqrt{1-x^6}} \cdot \int \frac{x dx}{\sqrt{1-x^6}} = \frac{\pi}{6} \tan \frac{\pi}{6}$.

VII. If $n = 3$, $k = 2$: $\int_0^1 \frac{x^4 dx}{\sqrt{1-x^6}} \cdot \int \frac{dx}{\sqrt{1-x^6}} = \frac{\pi}{12} \tan \frac{\pi}{3}$.

VIII. If $n = \frac{7}{2}$, $k = \frac{1}{2}$: $\int_0^1 \frac{x^3 dx}{\sqrt{1-x^7}} \cdot \int_0^1 \frac{x^2 dx}{\sqrt{1-x^7}} = \frac{2\pi}{7} \tan \frac{\pi}{14}$.

IX. If $n = \frac{7}{2}$, $k = \frac{3}{2}$: $\int_0^1 \frac{x^4 dx}{\sqrt{1-x^7}} \cdot \int_0^1 \frac{x dx}{\sqrt{1-x^7}} = \frac{2\pi}{21} \tan \frac{3\pi}{14}$.

X. If $n = \frac{7}{2}$, $k = \frac{5}{2}$: $\int_0^1 \frac{x^5 dx}{\sqrt{1-x^7}} \cdot \int \frac{dx}{\sqrt{1-x^7}} = \frac{2\pi}{35} \tan \frac{5\pi}{14}$.

XI. If $n = 4$, $k = 1$: $\int \frac{x^4 dx}{\sqrt{1-x^8}} \cdot \int \frac{x^2 dx}{\sqrt{1-x^8}} = \frac{\pi}{8} \tan \frac{\pi}{8}$.

XII. If $n = 4$, $k = 3$: $\int \frac{x^6 dx}{\sqrt{1-x^8}} \cdot \int \frac{dx}{\sqrt{1-x^8}} = \frac{\pi}{24} \tan \frac{3\pi}{8}$.

XIII. If $n = \frac{9}{2}$, $k = \frac{1}{2}$: $\int_0^1 \frac{x^4 dx}{\sqrt{1-x^9}} \cdot \int \frac{x^3 dx}{\sqrt{1-x^9}} = \frac{2\pi}{9} \tan \frac{\pi}{18}$.

XIV. If $n = \frac{9}{2}$, $k = \frac{5}{2}$: $\int_0^1 \frac{x^8 dx}{\sqrt{1-x^9}} \cdot \int \frac{x dx}{\sqrt{1-x^9}} = \frac{2\pi}{45} \tan \frac{5\pi}{18}$.

XV. If $n = \frac{9}{2}$, $k = \frac{7}{2}$: $\int_0^1 \frac{x^7 dx}{\sqrt{1-x^9}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^9}} = \frac{2\pi}{63} \tan \frac{7\pi}{18}$.

XVI. If $n = 5$, $k = 2$: $\int_0^1 \frac{x^6 dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^2 dx}{\sqrt{1-x^{10}}} = \frac{\pi}{20} \tan \frac{\pi}{5}$.

XVII. If $n = 5$, $k = 4$: $\int_0^1 \frac{x^8 dx}{\sqrt{1-x^{10}}} \cdot \int \frac{dx}{\sqrt{1-x^{10}}} = \frac{\pi}{40} \tan \frac{2\pi}{5}$.

XVIII. If $n = 6$, $k = 1$: $\int_0^1 \frac{x^6 dx}{\sqrt{1-x^{12}}} \cdot \int \frac{x^4 dx}{\sqrt{1-x^{12}}} = \frac{\pi}{12} \tan \frac{\pi}{12}$.

XIX. If $n = 6$, $k = 5$: $\int_0^1 \frac{x^{10} dx}{\sqrt{1-x^{12}}} \cdot \int \frac{dx}{\sqrt{1-x^{12}}} = \frac{\pi}{60} \tan \frac{5\pi}{12}$.

36. The formula

$$\int_0^1 \frac{y^{\lambda+\alpha-1} dy}{\sqrt{1-y^{2\alpha}}} \cdot \int_0^1 \frac{y^{\lambda-1} dy}{\sqrt{1-y^{2\alpha}}} = \frac{\pi}{2\lambda\alpha}, \tag{10}$$

is a special case of that found in *Section 24*. Replacing y with x, and then setting $\alpha = n$, we have:

$$\int_0^1 \frac{x^{\lambda+n-1} dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{\lambda-1} dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2\lambda n}.$$

Furthermore, from *Section 34* we have that

$$\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk} \tan\frac{k\pi}{2n},$$

and if we set $\lambda = k$:

$$\int_0^1 \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{k-1}dx}{\sqrt{1-x^{2n}}} = \tan\frac{k\pi}{2n},$$

and if we set $\lambda = n-k$:

$$\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{2n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{n-k}{k} \tan\frac{k\pi}{2n}.$$

37. Now, in order to put these equations in a more general setting, we set $v = 2$, $\mu = 1$ and make the substitution $y = x^{\frac{n}{\alpha}}$ in order to have the following from *section 24*:

$$\int_0^1 \frac{x^{\frac{\lambda n}{\alpha}+n-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{\frac{\lambda n}{\alpha}-1}dx}{\sqrt{1-x^{2n}}} = \frac{\alpha\pi}{2\lambda n^2}.$$

Let $\frac{\lambda n}{\alpha} = k$, and we will find the same formula that we found above, and the position $\frac{\lambda n}{\alpha} = n-k$ produces nothing new. Let's look at several particular cases:

1. Let $n = 1$, and $k = 0$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \div \int_0^1 \frac{xdx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

2. Let $n = \frac{3}{2}$, and $k = \frac{1}{2}$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} \div \int_0^1 \frac{dx}{x\sqrt{x(1-x^3)}} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

$$\int_0^1 \frac{xdx}{\sqrt{1-x^3}} \div \int_0^1 \frac{x^{\frac{3}{2}}dx}{\sqrt{1-x^3}} = 2\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}.$$

3. Let $n = 2$, and $k = 1$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \div \int_0^1 \frac{dx}{x\sqrt{1-x^4}} = \tan\frac{\pi}{4} = 1,$$

$$\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x^4}} \div \int_0^1 \frac{dx}{\sqrt{x(1-x^4)}} = \tan\frac{\pi}{8},$$

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \div \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} = \tan\frac{\pi}{4} = 1,$$

$$\int_0^1 \frac{x^{\frac{3}{2}}dx}{\sqrt{1-x^4}} \div \int_0^1 \frac{x^{\frac{5}{2}}dx}{\sqrt{1-x^4}} = 3\tan\frac{\pi}{8}.$$

And still some others:

$$\int_0^1 \frac{xdx}{\sqrt{1-x^5}} \div \int_0^1 \frac{dx}{\sqrt{x(1-x^5)}} = \tan\frac{\pi}{10},$$
$$\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x^5}} \div \int_0^1 \frac{dx}{\sqrt{1-x^5}} = \tan\frac{\pi}{5},$$
$$\int_0^1 \frac{xdx}{\sqrt{1-x^5}} \div \int_0^1 \frac{dx}{\sqrt{x(1-x^5)}} = \tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^{\frac{5}{2}}dx}{\sqrt{1-x^5}} \div \int_0^1 \frac{x^3dx}{\sqrt{1-x^5}} = \frac{3}{2}\tan\frac{\pi}{5},$$
$$\int_0^1 \frac{xdx}{\sqrt{1-x^6}} \div \int_0^1 \frac{dx}{\sqrt{1-x^6}} = \tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^6}} \div \int_0^1 \frac{x^4dx}{\sqrt{1-x^6}} = 2\tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^8}} \div \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \tan\frac{\pi}{8},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^8}} \div \int_0^1 \frac{x^6dx}{\sqrt{1-x^8}} = 3\tan\frac{\pi}{8},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^10}} \div \int_0^1 \frac{dx}{\sqrt{1-x^10}} = \tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^10}} \div \int_0^1 \frac{x^8dx}{\sqrt{1-x^10}} = 4\tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^10}} \div \int_0^1 \frac{xdx}{\sqrt{1-x^10}} = \tan\frac{\pi}{5},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^10}} \div \int_0^1 \frac{x^7dx}{\sqrt{1-x^10}} = \frac{3}{2}\tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^12}} \div \int_0^1 \frac{xdx}{\sqrt{1-x^12}} = \tan\frac{\pi}{12},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^12}} \div \int_0^1 \frac{x^{10}dx}{\sqrt{1-x^12}} = 5\tan\frac{\pi}{12},$$

38. These formulas are similar in form to those found in *Section 34*. All being members of the general family of integrals, $\int \frac{x^{m-1}dx}{\sqrt{1-x^n}}$. But those above were the product of two such integrals whose values I have erstwhile found. While in the immediate case we have instead the quotient, rather than the product, of two integrals. However in both cases, it is evident that the integration of one reduces to integration of the other. Since many of these reductions are entirely new, it is worth the effort to consider them more carefully. To this effect, I will list them in classes according to the exponent of the x which follows the radical sign. Therefore m and n being integers, we have:

I. Reducing formulas $\int \frac{x^{m-1}dx}{\sqrt{1-x^3}}$:

$$\int_0^1 \frac{xdx}{\sqrt{1-x^3}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{2\pi}{3} \tan\frac{\pi}{6} = \frac{2\pi}{3\sqrt{3}}.$$

II. Reducing formulas $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^4}}$:

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4} \tan\frac{\pi}{4} = \frac{\pi}{4}.$$

III. Reducing formulas $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^5}}$:

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^5}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{2\pi}{5} \tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^5}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^5}} = \frac{2\pi}{15} \tan\frac{3\pi}{10}.$$

IV. Reducing formulas $\int \frac{x^{m-1}dx}{\sqrt{1-x^6}}$:

$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^6}} = \frac{\pi}{6} \tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\pi}{12} \tan\frac{\pi}{3},$$
$$\int_0^1 \frac{xdx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^6}} = \tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{x^4dx}{\sqrt{1-x^6}} = 2\tan\frac{\pi}{6}.$$

V. Reduction of formulas $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^7}}$:

$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^7}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^7}} = \frac{2\pi}{7} \tan\frac{\pi}{14},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^7}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^7}} = \frac{2\pi}{21} \tan\frac{3\pi}{14},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^7}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^7}} = \frac{2\pi}{35} \tan\frac{5\pi}{14}.$$

VI. Reduction of formulas $\int \frac{x^{m-1}dx}{\sqrt{1-x^8}}$:

$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^8}} = \frac{\pi}{8}\tan\frac{\pi}{8},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^8}} = \frac{\pi}{16}\tan\frac{\pi}{4},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \frac{\pi}{24}\tan\frac{3\pi}{8},$$
$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \tan\frac{\pi}{8},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{x^6dx}{\sqrt{1-x^8}} = 3\tan\frac{\pi}{8}.$$

VII. Reduction of formulas $\int \frac{x^{m-1}dx}{\sqrt{1-x^9}}$:

$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^9}} \cdot \int_0^1 \frac{x^3dx}{\sqrt{1-x^9}} = \frac{2\pi}{9}\tan\frac{\pi}{18},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^9}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^9}} = \frac{2\pi}{27}\tan\frac{\pi}{6},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^9}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^9}} = \frac{2\pi}{45}\tan\frac{5\pi}{18},$$
$$\int_0^1 \frac{x^7dx}{\sqrt{1-x^9}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^9}} = \frac{2\pi}{63}\tan\frac{7\pi}{18}.$$

VIII. Reduction of formulas $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^10}}$:

$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^3dx}{\sqrt{1-x^{10}}} = \frac{\pi}{10}\tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^{10}}} = \frac{\pi}{20}\tan\frac{\pi}{5},$$
$$\int_0^1 \frac{x^7dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{10}}} = \frac{\pi}{30}\tan\frac{3\pi}{10},$$
$$\int_0^1 \frac{x^8dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{10}}} = \frac{\pi}{40}\tan\frac{2\pi}{5},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{10}}} = \tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^{10}}} = \tan\frac{\pi}{5},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^8dx}{\sqrt{1-x^{10}}} = 4\tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^8dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^7dx}{\sqrt{1-x^{10}}} = \frac{3}{2}\tan\frac{\pi}{5}.$$

39. Upon combining the quotients with the products from each class, one can form from them new products, which I will in general show. Given the product:

$$\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk}\tan\frac{k\pi}{2n},$$

as well as the two quotients:

I. $\int_0^1 \frac{x^{n-\alpha-1}dx}{\sqrt{1-x^{2n}}} \div \int_0^1 \frac{x^{\alpha-1}dx}{\sqrt{1-x^{2n}}} = \tan\frac{\alpha\pi}{2n}$.

II. $\int_0^1 \frac{x^{n+\beta-1}dx}{\sqrt{1-x^{2n}}} \div \int_0^1 \frac{x^{2n-\beta-1}dx}{\sqrt{1-x^{2n}}} = \frac{n-\beta}{\beta}\tan\frac{\beta\pi}{2n}$.

Then upon setting $\alpha = n-k$ and multiplying the first quotient against the product product, we will have:

$$\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk}.$$

Then, for the second quotient, setting $\beta = n-k$, and multiplying similarly, we obtain:

$$\int_0^1 \frac{x^{2n-k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2n(n-k)},$$

which does not differ from the preceding one. And so, for each class we have two products:

I. $\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{n-k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk} \tan \frac{k\pi}{2n}$,

II. $\int_0^1 \frac{x^{n+k-1}dx}{\sqrt{1-x^{2n}}} \cdot \int_0^1 \frac{x^{k-1}dx}{\sqrt{1-x^{2n}}} = \frac{\pi}{2nk}$,

of which the final one fits with those which I have already formerly demonstrated.

40. We develop these products for a few cases, where n and k are even integers, and we will have the following reductions for the case where $x = 1$:

I. Products of the form $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^4}}$:

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4} \tan \frac{\pi}{4} = \frac{\pi}{4}.$$

II. Products of the form $\int \frac{x^{m-1}dx}{\sqrt{1-x^6}}$:

$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^6}} = \frac{\pi}{6} \tan \frac{\pi}{6} = \frac{\pi}{6\sqrt{3}},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\pi}{12} \tan \frac{\pi}{3} = \frac{\pi}{4\sqrt{3}},$$
$$\int_0^1 \frac{x^3dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\pi}{6},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^6}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^6}} = \frac{\pi}{12}.$$

III. Products of the form $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^8}}$:

$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^8}} = \frac{\pi}{8} \tan \frac{\pi}{8},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^8}} = \frac{\pi}{16} \tan \frac{\pi}{4} = \frac{\pi}{16},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \frac{\pi}{24} \tan \frac{3\pi}{8},$$
$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \frac{\pi}{8},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^8}} = \frac{\pi}{16},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^8}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^8}} = \frac{\pi}{24}.$$

IV. Products of the form $\int_0^1 \frac{x^{m-1}dx}{\sqrt{1-x^{10}}}$:

$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^3dx}{\sqrt{1-x^{10}}} = \frac{\pi}{10}\tan\frac{\pi}{10},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^{10}}} = \frac{\pi}{20}\tan\frac{\pi}{5},$$
$$\int_0^1 \frac{x^7dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^{10}}} = \frac{\pi}{30}\tan\frac{3\pi}{10},$$
$$\int_0^1 \frac{x^8dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{10}}} = \frac{\pi}{40}\tan\frac{2\pi}{5},$$
$$\int_0^1 \frac{x^5dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{10}}} = \frac{\pi}{10},$$
$$\int_0^1 \frac{x^6dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{xdx}{\sqrt{1-x^{10}}} = \frac{\pi}{20},$$
$$\int_0^1 \frac{x^7dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^2dx}{\sqrt{1-x^{10}}} = \frac{\pi}{30},$$
$$\int_0^1 \frac{x^8dx}{\sqrt{1-x^{10}}} \cdot \int_0^1 \frac{x^3dx}{\sqrt{1-x^{10}}} = \frac{\pi}{40}.$$

41. After integrating these expressions, which are all specific realizations of the general formula

$$\int x^{m-1}(1-x^n)^k dx,$$

and which one could call algebraic, since dx is multiplied by an algebraic function of x, I pass, as I set to myself, to consider still several integrals where the differential dx is multiplied by a transcendent function of x, and of which the integral, in a certain way, can be expressed algebraically, or by the quadrature of the circle. These cases are even more remarkable, which shows us methods for evaluating them, and from the following observations will serve to be able to discover such methods.

42. I will not dwell upon the well known integral

$$\int_0^1 (\log\frac{1}{x})^n dx = n!,$$

which holds provided that n is a nonnegative integer. But when n is a fraction, the value of this integral is more difficult to find. As such, if $n = \frac{1}{2}$, I have demonstrated that the value of this integral is equal to $\frac{1}{2}\sqrt{\pi}$. From here, one easily derives these integrations which depend on them:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{2}}\,dx = \frac{1}{2}\sqrt{\pi},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{3}{2}}\,dx = \frac{1}{2}\frac{2}{2}\sqrt{\pi},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{5}{2}}\,dx = \frac{1}{2}\frac{3}{2}\frac{5}{2}\sqrt{\pi},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{7}{2}}\,dx = \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}\sqrt{\pi},$$

Since employing integration by parts, we have in general:

$$\int (\log\frac{1}{x})^m dx = x(\log\frac{1}{x})^m + m\int_0^1 (\log\frac{1}{x})^{m-1} dx;$$

therefore setting the bounds of the integral to be 0 and 1, we obtain:

$$\int_0^1 (\log\frac{1}{x})^m dx = m\int_0^1 (\log\frac{1}{x})^{m-1} dx.$$

43. This integration of the case $n = \frac{1}{2}$ can be expressed in this way:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{2}}\,dx = \sqrt{\frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}}$$

setting $x = 1$, and for the others, setting n to be a fraction, I have found the following reductions:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{3}} dx = \sqrt[3]{\frac{1}{3}\int_0^1 \frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int_0^1 \frac{xdx}{\sqrt[3]{(1-x^3)^2}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{3}} dx = 2\sqrt[3]{\frac{1}{3}\int_0^1 \frac{xdx}{\sqrt[3]{1-x^3}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[3]{1-x^3}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{4}} dx = \sqrt[4]{\frac{1}{4}\int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^3}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[4]{(1-x^4)^3}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{4}} dx = 2\sqrt[4]{\frac{1}{4}\int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^2}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[4]{(1-x^4)^2}} \cdot \int_0^1 \frac{x^5dx}{\sqrt[4]{(1-x^4)^2}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{3}{4}} dx = 3\sqrt[4]{\frac{1}{4}\int_0^1 \frac{x^2dx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{x^5dx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{x^8dx}{\sqrt[4]{1-x^4}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}} dx = 2\sqrt[5]{\frac{1}{5}\int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^4}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{5}} dx = 2\sqrt[5]{\frac{1}{5}\int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{x^5dx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{x^7dx}{\sqrt[5]{(1-x^5)^3}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{3}{5}} dx = 3\sqrt[5]{\frac{2}{5}\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{x^5dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{x^8dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{x^{11}dx}{\sqrt[5]{(1-x^5)^2}}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{4}{5}} dx = 4\sqrt[5]{\frac{6}{5}\int_0^1 \frac{x^3dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^7dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^{11}dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^{15}dx}{\sqrt[5]{1-x^5}}}.$$

44. Simplifying those integrals for which the numerator contains variables to higher or equal powers to that of the denominator, with the help of the reduction:

$$\int x^{m-1}(1-x^n)^k dx = \frac{m-n}{m+nk}\int x^{m-n-1}(1-x^n)^k dx,$$

we will find the following simplified forms:

$$\int_0^1 (\log \frac{1}{x})^{\frac{1}{2}} dx = \sqrt{\frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{1}{3}} dx = \sqrt[3]{\frac{1}{3} \int_0^1 \frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int_0^1 \frac{xdx}{\sqrt[3]{(1-x^3)^2}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{2}{3}} dx = 2\sqrt[3]{\frac{1}{9} \int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int_0^1 \frac{xdx}{\sqrt[3]{1-x^3}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{1}{4}} dx = \sqrt[4]{\frac{1}{4} \int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x^2dx}{\sqrt[4]{(1-x^4)^3}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{2}{4}} dx = 2\sqrt[4]{\frac{1}{16} \int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^2}} \cdot \int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^2}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[4]{(1-x^4)^2}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{5}{4}} dx = 3\sqrt[4]{\frac{2}{64} \int_0^1 \frac{dx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{xdx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[4]{1-x^4}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{1}{5}} dx = \sqrt[5]{\frac{1}{5} \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^4}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{2}{5}} dx = 2\sqrt[5]{\frac{1}{25} \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^3}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{3}{5}} dx = 3\sqrt[5]{\frac{2}{125} \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^2}}},$$

$$\int_0^1 (\log \frac{1}{x})^{\frac{4}{5}} dx = 4\sqrt[5]{\frac{6}{625} \int_0^1 \frac{dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^3dx}{\sqrt[5]{1-x^5}}}.$$

45. Therefore the values of the transcendent integral $\int_0^1 (\log \frac{1}{x})^n dx$, when n is a reduced fraction to values of the integrals, where dx is multiplied by an algebraic function of x. And yet, among these last formulas, there will remain a trigonometric part, since

$$\int \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^m}} = \frac{\pi}{n \sin \frac{m\pi}{n}}.$$

Then, in order to be better compose the other sets, setting in the formulas of Section 21: $2k = 2n + m - 2\lambda$ in order to have:

$$\int_0^1 \frac{x^{n-m-1}dx}{\sqrt[n]{(1-x^n)^{\lambda-m}}} \cdot \int_0^1 \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{\lambda}}} = \frac{\pi}{n(n-\lambda) \sin \frac{m\pi}{n}},$$

from here we will have:

I. If $n = 3$:

$$\int_0^1 \frac{xdx}{\sqrt[3]{1-x^3}} \cdot \int_0^1 \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sin\frac{\pi}{3}}.$$

II. If $n = 4$:

$$\int_0^1 \frac{x^2dx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^2}} = \frac{\pi}{8\sin\frac{\pi}{4}},$$

$$\int_0^1 \frac{x^2dx}{\sqrt[4]{(1-x^4)^2}} \cdot \int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{4\sin\frac{\pi}{4}},$$

$$\int_0^1 \frac{xdx}{\sqrt[4]{1-x^4}} \cdot \int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{4\sin\frac{\pi}{2}}.$$

III. If $n = 5$:

$$\int_0^1 \frac{x^3dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^2}} = \frac{\pi}{15\sin\frac{\pi}{5}},$$

$$\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^3}} = \frac{\pi}{10\sin\frac{\pi}{5}},$$

$$\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^3}} \cdot \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\pi}{5\sin\frac{\pi}{5}},$$

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^3}} = \frac{\pi}{10\sin\frac{2\pi}{5}},$$

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^2}} \cdot \int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^4}} = \frac{\pi}{5\sin\frac{2\pi}{5}},$$

$$\int_0^1 \frac{xdx}{\sqrt[5]{1-x^5}} \cdot \int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\pi}{5\sin\frac{3\pi}{5}}.$$

46. From here we see that multiplying all the formulas of the same order together, the product reduces to trigonometric expressions; as such we will have:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{2}} dx = \frac{1}{2}\sqrt{\pi},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{3}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{3}} dx = \frac{2}{9\sin\frac{\pi}{3}} \cdot \pi = \frac{2}{9}\sqrt{\frac{4\pi^2}{3}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{4}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{4}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{3}{4}} dx = \frac{6\pi\sqrt{\pi}}{4^3\sin\frac{\pi}{4}} = \frac{6}{4^3}\sqrt{\frac{8\pi^3}{4}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{5}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{3}{5}} dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{4}{5}} dx = \frac{24\pi^2}{5^4\sin\frac{\pi}{5}\sin\frac{2\pi}{5}} = \frac{24}{5^4}\sqrt{\frac{16\pi^4}{5}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{6}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{6}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{3}{6}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{4}{6}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{5}{6}}dx = \frac{120\pi^2\sqrt{\pi}}{6^5 \sin\frac{\pi}{6}\sin\frac{2\pi}{6}}$$
$$= \frac{120}{6^5}\sqrt{\frac{32\pi^5}{6}}.$$

From here we conclude that where will be in general:

$$\prod_{i=1}^{n-1}\int_0^1 (\log\frac{1}{x})^{\frac{i}{n}}dx = \frac{1\cdot 2\cdot 3\cdots(n-1)}{n^{n-1}}\sqrt{\frac{2^{n-1}\pi^{n-1}}{n}}, \tag{11}$$

a theorem which is quite worthy of attention.

47. The comparison of these formulas can be pushed even further, by considering this general theorem:

$$\int \frac{x^{\alpha-1}dx}{\sqrt[n]{(1-x^n)^{\beta}}} = \int \frac{x^{n-\beta-1}dx}{\sqrt[n]{(1-x^n)^{n-\alpha}}},$$

therefore the proceeding theorem is derived from Section 21. changes also into other forms. Then, the formulas of Section 29. furnishes the following comparisons:

$$\int_0^1 \frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}} \div \int_0^1 \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}} = \frac{\sin\frac{m\pi}{n}}{\sin\frac{k\pi}{n}},$$
$$\int_0^1 \frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{n+k-m}}} \div \int_0^1 \frac{x^{n-m-1}dx}{\sqrt[n]{(1-x^n)^{n+k-m}}} = \frac{\sin\frac{m\pi}{n}}{\sin\frac{k\pi}{n}},$$
$$\int_0^1 \frac{x^{n-k-1}dx}{\sqrt[n]{(1-x^n)^{n+m-k}}} \div \int_0^1 \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{n+k-m}}} = \frac{\sin\frac{m\pi}{n}}{\sin\frac{k\pi}{n}},$$
$$\int_0^1 \frac{x^{n-k-1}dx}{\sqrt[n]{(1-x^n)^{2n-m-k}}} \div \int_0^1 \frac{x^{n-m-1}dx}{\sqrt[n]{(1-x^n)^{n+k-m}}} = \frac{\sin\frac{m\pi}{n}}{\sin\frac{k\pi}{n}}.$$

of which the last reduces to the first, since in place of m and k one can put $n-m$ and $n-k$.

48. Now, since

$$\int \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}} = \frac{n-k}{m}\int \frac{x^{m+n-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}},$$

one will then have this comparison:

$$\int \frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}} : \int \frac{x^{m+n-1}dx}{\sqrt[n]{(1-x^n)^{m+k}}} = \frac{n-k}{m}\sin\frac{m\pi}{n} : \sin\frac{k\pi}{n}$$

upon taking m for a negative number:

$$\int \frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{k-m}}} : \int \frac{x^{n-m-1}dx}{\sqrt[n]{(1-x^n)^{k-m}}} = \frac{n-k}{m} \sin \frac{m\pi}{n} : \sin \frac{k\pi}{n},$$

from here we draw the particular following comparisons:

$$\int_0^1 \frac{xdx}{\sqrt[4]{(1-x^4)^3}} \div \int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{\sin \frac{\pi}{4}}{\sin \frac{\pi}{2}} = \frac{1}{\sqrt{2}},$$

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} \div \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\sin \frac{\pi}{5}}{\sin \frac{2\pi}{5}},$$

$$\int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^3}} \div \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^3}} = \frac{\sin \frac{\pi}{5}}{\sin \frac{2\pi}{5}},$$

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^2}} \div \int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^2}} = \frac{2\sin \frac{\pi}{5}}{\sin \frac{2\pi}{5}},$$

$$\int_0^1 \frac{xdx}{\sqrt[5]{1-x^5}} \div \int_0^1 \frac{x^3dx}{\sqrt[5]{1-x^5}} = \frac{3\sin \frac{\pi}{5}}{\sin \frac{2\pi}{5}}.$$

49. In order to make obvious the usage of these reductions, we will consider the particular formulas which enter in the expressions of the formulas

$$\int_0^1 (\log \frac{1}{x})^{\frac{1}{5}} dx; \quad \int_0^1 (\log \frac{1}{x})^{\frac{2}{5}} dx; \quad \int_0^1 (\log \frac{1}{x})^{\frac{3}{5}} dx; \quad \int_0^1 (\log \frac{1}{x})^{\frac{4}{5}} dx,$$

and first the number of all the said formulas being 16, there will be 4 which depend on the quadrature of the circle.

$$\int_0^1 \frac{dx}{\sqrt[5]{1-x^5}} = \frac{\pi}{5\sin \frac{\pi}{5}},$$

$$\int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^2}} = \frac{\pi}{5\sin \frac{2\pi}{5}},$$

$$\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\pi}{5\sin \frac{\pi}{5}}.$$

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^3}} = \frac{\pi}{5\sin \frac{3\pi}{5}} = \frac{\pi}{5\sin \frac{2\pi}{5}},$$

For the other 12 of the general furnished reduction:

$$\int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^4}} = \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^3}},$$
$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} = \int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^2}},$$
$$\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^3}} = \int_0^1 \frac{xdx}{\sqrt[5]{1-x^5}},$$
$$\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^2}} = \int_0^1 \frac{x^2dx}{\sqrt[5]{1-x^5}}.$$

Then we come to find:

$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}\int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}},$$
$$\int_0^1 \frac{xdx}{\sqrt[5]{(1-x^5)^3}} = \frac{\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}\int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^3}},$$
$$\int_0^1 \frac{x^2dx}{\sqrt[5]{(1-x^5)^2}} = \frac{2\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}\int_0^1 \frac{x^3dx}{\sqrt[5]{(1-x^5)^2}},$$
$$\int_0^1 \frac{xdx}{\sqrt[5]{1-x^5}} = \frac{2\sin\frac{1}{5}\pi}{\sin\frac{2}{5}\pi}\int_0^1 \frac{x^3dx}{\sqrt[5]{1-x^5}},$$

those which one can add to the products of two such formulas quotients in Section 45. for the case where $n = 5$.

50. If we examine the integral equalities in the previous section, we find that all twelve reduce to only two. In the interest of abbreviation, set

$$y = \frac{1}{\sqrt[5]{1-x^5}}, \quad \alpha = \sin\frac{\pi}{5}, \quad \beta = \sin\frac{2\pi}{5},$$

then we can write these integrals purely as a combination trigonometric functions and the two simple integrals $\int y^2dx$ and $\int y^3dx$:

$$\int_0^1 y^4dx = \frac{\beta}{\alpha}\int y^2dx \quad \int_0^1 xy^4dx = \int_0^1 y^3dx \quad \int_0^1 x^2y^4dx = \int_0^1 y^2dx,$$

$$\int_0^1 y^3dx = \frac{\pi}{5\alpha} \quad \int_0^1 x^3ydx = \frac{\pi}{15\alpha\int y^2dx} \quad \int_0^1 x^2y^3dx = \frac{\pi}{5\beta\int y^3dx},$$

$$\int_0^1 ydx = \frac{\alpha}{\beta}\int_0^1 y^3dx \quad \int_0^1 xy^2dx = \frac{\pi}{5\beta} \quad \int_0^1 x^3y^3dx = \frac{\pi}{5\beta\int_0^1 y^2dx},$$

$$\int_0^1 x^2y^3dx = \frac{\pi}{5\beta} \quad \int_0^1 x^2ydx = \frac{\pi}{10\alpha\int y^3dx} \quad \int_0^1 x^3y^2dx = \frac{\pi}{10\alpha\int y^3dx},$$

$$\int_0^1 x^3y^4dx = \frac{\pi}{5\alpha} \quad \int_0^1 xydx = \frac{\pi}{5\beta\int y^2dx}$$

Therefore let $\int y^2dx = A$ and $\int y^3dx = B$, and the values of our transcendent formulas will be:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}}dx = \sqrt[5]{\frac{\beta\pi A^2B}{5^2\alpha^2}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx = 2\sqrt[5]{\frac{\alpha\pi^2B^2}{5^4\beta^3A}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{3}{5}}dx = 3\sqrt[5]{\frac{\pi^3A}{5^6\alpha\beta^2B^2}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{4}{5}}dx = 4\sqrt[5]{\frac{\pi^4}{5^8\alpha^3\beta A^2B}}.$$

51. From here we see that not only the product of all these four formulas depend uniquely on the quadrature of the circle, but also the product of two, of which the exponents make together unity, to know:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{4}{5}}dx = \frac{4\pi}{5^2\sin\frac{\pi}{5}},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx = \frac{6\pi}{5^2\sin\frac{2\pi}{5}}.$$

Besides this, we can deduce from them the following equalities:

$$\Big[\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}}dx\Big]^2 \div \int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)^4}} = \frac{\beta A}{2\alpha},$$

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}}dx \cdot \Big[\int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx\Big]^2 = \frac{4\pi B}{5^2\sin\frac{2\pi}{5}}\int_0^1 \frac{dx}{\sqrt[5]{(1-x^5)3}}.$$

52. If we join these previous determinations, we can draw from them the following general conclusions:

$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{2}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{1}{2}}dx = \frac{\pi}{2^2 \sin\frac{\pi}{2}},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{3}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{2}{3}}dx = \frac{2\pi}{3^2 \sin\frac{\pi}{3}},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{4}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{3}{4}}dx = \frac{3\pi}{4^2 \sin\frac{\pi}{4}},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{1}{5}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{4}{5}}dx = \frac{4\pi}{5^2 \sin\frac{\pi}{5}},$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{2}{5}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{3}{5}}dx = \frac{6\pi}{5^2 \sin\frac{2\pi}{5}},$$

and in general:

$$\int_0^1 (\log\frac{1}{x})^{\frac{m}{n}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{n-m}{n}}dx = \frac{m(n-m)\pi}{n^2 \sin\frac{m\pi}{n}},$$

therefore since

$$\int_0^1 (\log\frac{1}{x})^{\frac{n-m}{n}}dx = \frac{n-m}{n}\int_0^1 (\log\frac{1}{x})^{\frac{-m}{n}}dx,$$

we will have:

$$\int_0^1 (\log\frac{1}{x})^{\frac{m}{n}}dx \cdot \int_0^1 (\log\frac{1}{x})^{\frac{-m}{n}}dx = \frac{m\pi}{n \sin\frac{m\pi}{n}} = m\int_0^1 \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^m}}.$$

53. This last equality can easily be immediately demonstrated by developing the most simple case, where the exponent is an even number:

$$\int_0^1 (\log\frac{1}{x})^{\lambda}dx = 1\cdot 2\cdot 3\cdots\lambda.$$

And yet, this finite expression can be expressed by an infinite product, as:

$$\int (\log\frac{1}{x})^{\lambda}dx = (\frac{2}{1})^{\lambda}\cdot\frac{1}{1+\lambda}\cdot(\frac{3}{2})^{\lambda}\cdot\frac{2}{2+\lambda}\cdot(\frac{4}{3})^{\lambda}\cdot\frac{3}{3+\lambda}\cdots\cdot.$$

Now letting $\lambda = \frac{m}{n}$ to have:

$$\int (\log\frac{1}{x})^{\frac{m}{n}}dx = (\frac{2}{1})^{\frac{m}{n}}\cdot\frac{n}{n+m}\cdot(\frac{3}{2})^{\frac{m}{n}}\cdot\frac{2n}{2n+m}\cdot(\frac{4}{3})^{\frac{m}{n}}\cdot\frac{3n}{3n+m}\cdots\cdot,$$

and also for m negative:

$$\int (\log\frac{1}{x})^{\frac{-m}{n}}dx = (\frac{2}{1})^{\frac{-m}{n}}\cdot\frac{n}{n-m}\cdot(\frac{3}{2})^{\frac{-m}{n}}\cdot\frac{2n}{2n-m}\cdot(\frac{4}{3})^{\frac{-m}{n}}\cdot\frac{3n}{3n-m}\cdots\cdot.$$

The product of these two formulas evidently give:

$$\frac{n^2}{n^2-m^2}\cdot\frac{4n^2}{4n^2-m^2}\cdot\frac{9n^2}{9n^2-m^2}\ \text{etc.} = \frac{m\pi}{n\sin\frac{m\pi}{n}}.$$

54. We could set farther these researches, since

$$\int_0^1 (\log\frac{1}{x})^{\frac{p}{n}}dx = (\frac{2}{1})^{\frac{p}{n}}\cdot\frac{n}{n+p}\cdot(\frac{3}{2})^{\frac{p}{n}}\cdot\frac{2n}{2n+p}\cdots\cdot,$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{q}{n}}dx = (\frac{2}{1})^{\frac{q}{n}}\cdot\frac{n}{n+q}\cdot(\frac{3}{2})^{\frac{q}{n}}\cdot\frac{2n}{2n+q}\cdots\cdot,$$
$$\int_0^1 (\log\frac{1}{x})^{\frac{p+q}{n}}dx = (\frac{2}{1})^{\frac{p+q}{n}}\cdot\frac{n}{n+p+q}\cdot(\frac{3}{2})^{\frac{p+q}{n}}\cdot\frac{2n}{2n+p+q}\cdots\cdot.$$

The product of the two first divided by the last, gives:

$$\frac{\int(\log\frac{1}{x})^{\frac{p}{n}}dx\cdot\int(\log\frac{1}{x})^{\frac{q}{n}}dx}{\int(\log\frac{1}{x})^{\frac{p+q}{n}}dx} = \frac{n(n+p+q)}{(n+p)(n+q)}\cdot\frac{2n(2n+p+q)}{(2n+p)(2n+q)}\cdots\cdot,$$

of which the value is

$$q\int\frac{x^{n+p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{pq}{p+q}\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{pq}{p+q}\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}},$$

or in other words, it equals:

$$q\int x^{q-1}\sqrt[n]{(1-x^n)^p}dx = p\int x^{p-1}\sqrt[n]{(1-x^n)^q}dx,$$

formula which conduces to the proceeding one, when one sets $p=m$ and $q=-m$. In the same way, one will find the value of

$$\frac{\int(\log\frac{1}{x})^{\frac{p}{n}}dx\cdot\int(\log\frac{1}{x})^{\frac{q}{n}}dx\cdot\int(\log\frac{1}{x})^{\frac{r}{n}}dx}{\int(\log\frac{1}{x})^{\frac{p+q+r}{n}}dx} = \frac{pqr}{p+q+r}\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}.$$

55. Finally, in order to finish this paper, the summation of the reciprocal series of powers we furnish still the value of the following transcendent formulas, when one sets after integration $x=1$:

$$\int\frac{1}{x}\log\frac{1}{1-x}dx = \frac{\pi^2}{6};\quad \int\frac{1}{x}\log(1+x)dx = \frac{\pi^2}{12},$$
$$\text{and}\quad \int\frac{1}{x}\log\sqrt{\frac{1+x}{1-x}}dx = \frac{\pi^2}{8},$$

and these others, more composed:

$$\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{1}{x}\log\frac{1}{1-x}dx = \frac{\pi^4}{90};\quad \int\frac{dx}{x}\int\frac{dx}{x}\int\frac{1}{x}\log(1+x)dx = \frac{7\pi^4}{720},$$
$$\int\frac{dx}{x}\int\frac{dx}{x}\arctan x = \frac{\pi^3}{32};\quad \int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\log\sqrt{\frac{1+x}{1-x}} = \frac{\pi^4}{96}.$$

And now, there does not seem any direct rout which we can travel along to these determinations, which merits itself, even more attention.