Synopsis by section of Euler's paper E796,

Recherches sur le problem de trois nombres carres tels que la somme de deux quelconques moins le troisieme fasse un nombre carre

"Research into the problem of three square numbers such that the sum of any two less the third one provides a square number."

Kristen McKeen and Thomas J. Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

1. Euler begins by stating the problem to be considered in this paper: : Let x, y, z be positive integers such that the sum of any two minus the third is a perfect square. In other words we require that the following three equations be satisfied

$$y^2 + z^2 - x^2 = p^2,$$
$$x^2 + z^2 - y^2 = q^2, \qquad (1)$$
$$x^2 + y^2 - z^2 = r^2.$$

By adding these equations he finds that this problem is equivalent to the second problem:

Find three square numbers such that half the sum of any two of them is also a square

$$\frac{p^2 + q^2}{2} = z^2, \frac{p^2 + r^2}{2} = y^2 \text{ and } \frac{q^2 + r^2}{2} = x^2.$$

While he does not say it, Euler avoids trivial solutions in which two of the numbers x, y, z are equal.

2. If we find a solution of (1) as x, y, z, then all their multiples nx, ny, nz will also be a solution. To avoid these trivial solutions, in the following we will search for three

numbers, having no common divisor. With this restriction, Euler shows that all three

numbers must be odd.

3.   Euler digresses in this section , and explores a generalization of his main problem.

He shows that if we want to determine four square numbers such that the sum of three

less the fourth one is a perfect square, no solution is possible.

4.   After giving extensive motivation, he finds that by taking

$$x = b^2 + ab - a^2, \ y = a^2 + ab - b^2 \text{ and } z = a^2 + b^2,$$

then the first two equations of (1) will be satisfied. He must now determine values of a

and b which also satisfy the third equation which is

$$x^2 + y^2 - z^2 = a^4 + b^4 - 4a^2b^2 = r^2.$$

5.   Euler looks for a solution to $a^4 + b^4 - 4a^2b^2 = r^2$. He remarks that an easy

solution is $a = 2b$. To find another solution Euler tries $a = b(\zeta + 2),$ (Euler uses z rather

than our ζ , which could be confusing) and after some calculation shows that

$$y^2 - x^2 = 4ab(a^2 - b^2),\qquad\qquad(2)$$

and determines that $\zeta = \dfrac{-23}{4}$, which leads to $a = -\dfrac{15b}{4}$. Thus he has his first solution

$\qquad x = 149, \ y = 269, \ z = 241,$

which appears to be the smallest numbers possible. From this, he finds

$\qquad p = 329, \ q = 89, \text{ and } r = 191.$

6.   He now tries different methods of factoring the right side of (2). He first tries

$y + x = \dfrac{2m}{n} ab$ and $y - x = \dfrac{2n}{m}(a^2 - b^2)$, but, without showing calculations remarks that this fails. He then tries $y + x = 2a(a+b),$ and $y - x = 2b(a-b)$, and after some calculations shows that this fails also.

7.   He has tried other methods of this type, and reveals that after much long and difficult calculation he has had no success. He will now turn to four other "absolutely remarkable methods" which "without difficulty" yield general formulas for x, y, and z that produce an infinite number of solutions.

Easy methods for finding more general solutions.

First method.

8.   Euler sets $s = x^2 + y^2 + z^2$, and equations (1) become

$$s - 2x^2 = p^2, \text{ or } s = p^2 + 2x^2,$$

$$s - 2y^2 = q^2, \text{ or } s = q^2 + 2y^2, \tag{3}$$

$$s - 2z^2 = r^2, \text{ or } s = r^2 + 2z^2.$$

Thus s has to be, in three different ways, the sum of a square plus twice a square.

9.   He shows that if s is a prime number that can be expressed in the form $s = a^2 + 2b^2$, then this form is unique. That is to say, there are no numbers c and d, different from a and b, such that $s = c^2 + 2d^2$. It follows immediately from (3) that s cannot be prime.

10.   He proposes to demonstrate that if s has the form $a^2 + 2b^2$ and satisfies (3) then it must be the product of at least three prime factors.

11.   Euler states without proof that all odd numbers of the form $a^2 + 2b^2$ are always of the form $8n+1$ or $8n+3$, and that when the number is even and of the form $a^2 + 2b^2$, it is twice one or the other of the two formulas.

12.   He gives a list of primes of the form $8n+1$ and $8n+3$ and shows that all of them can be expressed in the form $a^2 + 2b^2$. He calls this remarkable.

$$
\begin{array}{ll}
8n+1 & 8n+3 \\
17 = 3^2 + 2\cdot 2^2 & 3 = 1^2 + 2\cdot 1^2 \\
41 = 3^2 + 2\cdot 4^2 & 11 = 3^2 + 2\cdot 1^2 \\
73 = 1^2 + 2\cdot 6^2 & 19 = 1^2 + 2\cdot 3^2 \\
89 = 9^2 + 2\cdot 2^2 & 43 = 5^2 + 2\cdot 3^2 \\
97 = 5^2 + 2\cdot 6^2 & 59 = 3^2 + 2\cdot 5^2 \\
113 = 9^2 + 2\cdot 4^2 & 67 = 7^2 + 2\cdot 3^2 \\
137 = 3^2 + 2\cdot 8^2 & 83 = 9^2 + 2\cdot 1^2 \\
 & 107 = 3^2 + 2\cdot 7^2 \\
 & 131 = 9^2 + 2\cdot 5^2 \\
 & 139 = 11^2 + 2\cdot 3^2 \\
\end{array}
$$

We used a computer to extend Euler's list to all primes congruent to 1 or 3 mod 8 less than 800,000. In all cases these primes were of the form $a^2 + 2b^2$. Is Euler conjecturing that this is true for all such primes?

13.   Euler states Fermat's little theorem: *If $2m+1$ is prime and not a divisor of c, then $2m+1$ divides $c^{2m} - 1$.* He uses it to show that if $8n+1$ is prime, then it divides some numbers of the form $a^2 + 2b^2$.

14.   He again uses Fermat's little theorem to show that if $8n + 3$ is prime, then it also divides some numbers of the form $a^2 + 2b^2$.

(It appears as though Euler suspects that all primes of the form 8n+1 and 8n+3 are of the form $a^2 + 2b^2$, but he is only able to show that these primes divide some number of the form $a^2 + 2b^2$.)

Now he returns to solving equations (3). We have seen that s must have at least three factors and he tries

$$s = (a^2 + 2b^2)(c^2 + 2d^2)(f^2 + 2g^2).$$

He then sets $(a^2 + 2b^2)(c^2 + 2d^2) = m^2 + 2n^2$ from which it follows that

$$m = ac \pm 2bd, \quad n = bc \mp ad. \tag{4}$$

(It is an elementary exercise to show that numbers of the form $a^2 + 2b^2$ are closed under multiplication.) Our sum s can be expressed as $s = (m^2 + 2n^2)(f^2 + 2g^2) = \zeta^2 + 2v^2$ and we will have similarly

$$\zeta = mf \pm 2ng \text{ and } v = nf \mp mg. \tag{5}$$

(Euler uses z rather than ζ in the above which is a confusing choice of variable. Thus we use ζ for clarity.)

15.   Using (4) we eliminate m and n from (5). This gives us four expressions for ζ and four for v. For ζ we get

1) $f(ac + 2bd) + 2g(bc - ad)$,

2) $f(ac + 2bd) - 2g(bc - ad)$,

3) $f(ac - 2bd) + 2g(bc + ad)$,

4) $f(ac-2bd)-2g(bc+ad),$

and for v:

1) $f(bc-ad)-g(ac+2bd),$

2) $f(bc-ad)+g(ac+2bd),$

3) $f(bc+ad)-g(ac-2bd),$

4) $f(bc+ad)+g(ac-2bd).$

16.   While we have four different values for ζ and $v,$ we require only three, because

of the conditions $s=p^2+2x^2,\ s=q^2+2y^2$ and $s=r^2+2z$. Using the first three values

of ζ and v listed above we get

$$f(ac+2bd)+2g(bc-ad)=p,$$
$$f(ac+2bd)-2g(bc-ad)=q,$$
$$f(ac-2bd)+2g(bc+ad)=r,$$
$$f(bc-ad)-g(ac+2bd)=x,$$
$$f(bc-ad)+g(ac+2bd)=y,$$
$$f(bc+ad)-g(ac-2bd)=z.$$

17.   Using the above values for $x,\ y,\ z,$ Euler calculates $s=x^2+y^2+z^2$ and obtains the

expression $s=Af^2+Bg^2+2Cfg,$ where

$$A=3b^2c^2-2abcd+3a^2d^2,$$
$$B=3a^2c^2+4abcd+12b^2d^2,$$
$$C=-(bc+ad)(ac-2bd).$$

He now compares the above expression with

$$s=(a^2+2b^2)(c^2+2d^2)(f^2+2g^2)$$

$$=ff(a^2c^2+2b^2c^2+2a^2d^2+4b^2d^2)+2g^2(a^2c^2+2b^2c^2+2a^2d^2+4b^2d^2).$$

To make these two expressions for s equal he sets

$$Ff^2 + Gg^2 + 2Cfg = 0, \tag{6}$$

where

$$F = b^2c^2 - 2abcd + a^2d^2 - a^2c^2 - 4b^2d^2,$$
$$G = a^2c^2 + 4abcd + 4b^2d^2 - 4b^2c^2 - 4a^2d^2,$$
$$C = -(bc + ad)(ac - 2bd).$$

We must find values of the six numbers a, b, c, d, f, g, which satisfy (6) and from these

we can find x, y, z and also p, q, r. We note that (6) is an important quadratic equation for

$\dfrac{f}{g}$

$$F\left(\frac{f}{g}\right)^2 + 2C\frac{f}{g} + G = 0. \tag{7}$$

Euler will use this equation several times in the remainder of the paper.

18.   By taking $F = 0$, Euler simplifies the solution of (7) and gets $\dfrac{f}{g} = -\dfrac{G}{2C}$. After

reducing $-\dfrac{G}{2C}$ to smaller terms, Euler will take the numerator for f and the denominator

for g, and all the expressions above will b rational numbers.

19.   The value $F = b^2c^2 - 2abcd + a^2d^2 - a^2c^2 - 4b^2d^2$ factors as

$F = \{(b + a)c + (a + 2b)d\}\{(b - a)c + (a - 2b)d\}$. Since $F = 0$, This leads to the two

solutions

$$\frac{c}{d} = \frac{-a - 2b}{b + a} \quad \text{and} \quad \frac{c}{d} = \frac{2b - a}{b - a}.$$

20.   In the same manner Euler tries making the value of G vanish. In this case he

gets $\dfrac{c}{d} = \dfrac{2b + 2a}{a + 2b}$, or $\dfrac{c}{d} = \dfrac{2b - 2a}{a - 2b}$, but these values do not result in new solutions.

21.   Having found a general solution to his original problem, Euler now reviews the

details of his method for finding numerical values:

Summary of Euler's first method

1) Begin with any two numbers a and b, then find c and d by one or the other

of the two formulas $\dfrac{c}{d} = \dfrac{-a-2b}{b+a}$, or $\dfrac{c}{d} = \dfrac{2b-a}{b-a}$.

2) Next we calculate

$$\frac{(a^2 - 4b^2)c^2 + 4(b^2 - a^2)d^2 + 4abcd}{2(bc + ad)(ac - 2bd)} .$$

After reducing this fraction to lowest terms, we take f equal to the numerator, and g equal

to the denominator.

3) Now we find of x, y, z by the formulas

$$x = f(bc - ad) - g(ac + 2bd),$$
$$y = f(bc - ad) + g(ac + 2bd),$$
$$z = f(bc + ad) - g(ac - 2bd).$$

4) Finally the letters p, q, r are also found from

$$p = f(ac + 2bd) + 2g(bc - ad),$$
$$q = f(ac + 2bd) - 2g(bc - ad),$$
$$r = f(ac - 2bd) + 2g(bc + ad).$$

Euler ends this section by showing the detailed calculations in three examples.

Example 1. Let $a = 1$ and $b = 1$, and get for *x, y, z* the numbers 241, 269 and 149, and for

p, q, r he obtains -191, 89 and 329. Also $s = 3 \cdot 17 \cdot 41 \cdot 73$.

Example 2. Let $a = 1$ and $b = 2$, and get two cases. In the first case he obtains for *x, y, z* the values 397, 593 and 707. For *p, q, r* he gets 833, 553 and 97. Also $s = 9 \cdot 11 \cdot 10193$.

In the second case he obtains for *x, y, z* the values 3365, 6697 and 6755. For *p, q, r* he gets -8897, 3479 and 3247. Also $s = 9 \cdot 43 \cdot 263057$.

Example 3. Let 3 and 1 and obtain for *x, y, z* values 8405, 12913 and -11795. For *p, q, r* he gets 15337, -6559 and 9913. Also $s = 11 \cdot 57 \cdot 600497$.

22.   Euler lists a few additional results involving relatively small numbers.

	I	II	III	IV	V
$x =$	241	397	425	595	493
$y =$	269	593	373	769	797
$z =$	149	707	205	965	937
$p =$	191	833	23	1081	1127
$q =$	89	553	289	833	697
$r =$	329	97	527	119	289

Second Method.

23.   In section 17, the solution of our problem was reduced to the quadratic equation

(7)

$$F\left(\frac{f}{g}\right)^2 + 2C\frac{f}{g} + G = 0$$

where

$$F = b^2c^2 - 2abcd + a^2d^2 - a^2c^2 - 4b^2d^2,$$
$$G = a^2c^2 + 4abcd + 4b^2d^2 - 4b^2c^2 - 4a^2d^2,$$
$$C = -(bc + ad)(ac - 2bd).$$

Rather than set $F = 0$, Euler now examines the general solution of the quadratic

$$\frac{f}{g} = \frac{-C \pm \sqrt{C^2 - FG}}{F}. \text{ Let } V^2 = C^2 - FG \text{ so that } \frac{f}{g} = \frac{-C \pm V}{F}. \text{ By substituting the values}$$

of *C*, *F*, and *G*, and letting $m = \frac{ab}{a^2 - 2b^2}$ we get the expression

$$\frac{V^2}{(a^2 - 2b^2)^2} = c^4 + 8mc^3d - 4c^2d^2 - 16mcd^3 + 4d^4.$$

24.   Since this formula must be a square, let us assume its square root is equal to

$$\frac{V}{a^2 - 2b^2} = c^2 - 4mcd + 2d^2 \quad .$$

To make this true Euler finds that he needs $\frac{c}{d} = \frac{2m^2 + 1}{2m}$. Thus, he takes $c = 2m^2 + 1$ and

$d = 2m$, and gets

$$\frac{V}{a^2 - 2b^2} = 4m^2 + 1 - 12m^4.$$

25.   We now summarize Euler's second method of solution in more detail than given

in his paper.

Summary of Euler's second method

1)   Select arbitrary numbers *a* and *b*. Next calculate $m = ab / (a^2 - 2b^2)$, and

$c = 2m^2 + 1$ and $d = 2m$. Find $V = \left(a^2 - 2b^2\right)\left(4m^2 + 1 - 12m^4\right)$.

2)   Next find *F* and *C* from (*same as first solution*)

$$F = b^2c^2 - 2abcd + a^2d^2 - a^2c^2 - 4b^2d^2,$$
$$C = -(bc + ad)(ac - 2bd).$$

3) Find *f* and *g* from $\frac{f}{g} = \frac{-C \pm V}{F}$.

4) Now we find x, y, z by the formulas (*same as first solution*)

$$x = f(bc - ad) - g(ac + 2bd),$$
$$y = f(bc - ad) + g(ac + 2bd),$$
$$z = f(bc + ad) - g(ac - 2bd).$$

5) Finally the numbers p, q, r are also found from (*same as first solution*)

$$p = f(ac + 2bd) + 2g(bc - ad),$$
$$q = f(ac + 2bd) - 2g(bc - ad),$$
$$r = f(ac - 2bd) + 2g(bc + ad).$$

Euler gives two examples:

Example 1. Let $a = 2$ and $b = 1$, then $m = 1$, $c = 3$, $d = 2$, $f = 28$, $g = 51$, and

get $x = 482$, $y = -538$, $z = 298$, and $p = 382$, $q = 178$, $r = -658$.

Example 2. Let $a = 3$ and $b = 2$, then $m = 6$, $c = 73$, $d = 12$, $f = -7$,

$g = 17$, and obtain $x = 5309$, $y = 3769$, $z = 4181$, and $p = 1871$, $q = 5609$, $r = 4991$.

Third method

26.   Starting with $s = (a^2 + 2b^2)(c^2 + 2d^2)(f^2 + 2g^2)$, Euler assumes that the first

factor can be expressed in two different ways as $\alpha^2 + 2\beta^2 = a^2 + 2b^2$. Euler uses a and b

to calculate x, y, p, q, while α and β are used to find z and r. He gets

$$x = f(bc - ad) - g(ac + 2bd), \qquad\qquad p = f(ac + 2bd) + 2g(bc - ad),$$

$$y = f(bc - ad) + g(ac + 2bd), \qquad\qquad q = f(ac + 2bd) - 2g(bc - ad),$$

$$z = f(\beta c + \alpha d) - g(\alpha c + 2\beta d), \qquad\qquad r = f(\alpha c - 2\beta d) + 2g(\beta c + \alpha d).$$

After more manipulation which we will skip in this summary he again gets the important

quadratic (7)

$$F\left(\frac{f}{g}\right)^2 + 2C\frac{f}{g} + G = 0,$$

now with

$$C = (\alpha c - 2\beta d)(\beta c + \alpha d),$$

$$F = \big((\beta + a)c + (\alpha + 2b)d\big)\big((\beta - a)c + (\alpha - 2b)d\big),$$

$$G = \big((\alpha + 2b)c - 2(\beta + a)d\big)\big((\alpha - 2b)c - 2(\beta - \alpha)d\big).$$

27.   To solve the above quadratic he selects the simple solution where $F = 0,$ by

setting

$$\frac{c}{d} = \frac{-\alpha - 2b}{\beta + a} \quad \text{or} \quad = \frac{-\alpha + 2b}{\beta - a}.$$

Thus $\dfrac{f}{g} = \dfrac{G}{2C}$ from which Euler gets $f = (aa + 2bb)(cc + 2dd)$

and $g = -2(\alpha c - 2\beta d)(\beta c + \alpha d).$

28.   Euler states that he will simplify the calculations in a "rule" in the next section.

Summary of Euler's third method

29.   Arbitrarily select two numbers m and n, in which m must be odd, and compute

$s = m^2 + 2n^2,\ \ t = m^2 - 2n^2$ and $u = 2mn$. (Here Euler uses the letter s, but it is not the

sum of the squares that he uses so often previously.) Now the solution is

$$x = s(s+u)(3s+4u) - 2t^2(s+2u), \qquad p = st(3s+4u) + 4t(s+u)(s+2u),$$

$$y = s(s+u)(3s+4u) + 2t^2(s+2u), \qquad q = st(3s+4u) - 4t(s+u)(s+2u),$$

$$z = st(3s+4u) + 2t(s+2u)^2, \qquad r = s(s+2u)(3s+4u) - 4t^2(s+2u).$$

30.   Each pair of numbers m and n gives, two different solutions, depending on

whether we take m and n positive or negative. Here are some examples.

Example 1. Let $m = 1$ and $n = \pm 1$; then $s = 3$, $t = 1$, $u = \pm 2$. First let $u = -2$,

and get $s + u = 1$, $s + 2u = -1$, $3s + 4u = 1$ and, consequently,

$$x = 5, \ y = 1, \ z = 5, \ p = -1, \ q = 7, \ r = 1.$$

Since $x = z$ this solution is rejected. Next Euler selects $u = 2$ and gets

$$x = 241, \ y = 269, \ z = 149, \ p = 191, \ q = -89, \ r = 329.$$

Example 2. Let, in this example, $m = 1$ and $n = 2$; then $s = 9$, $t = -7$, $u = \pm 4$.

First he takes $u = -4$; and get $s + u = 5$, $s + 2u = 1$, $3s + 4u = 11$. So

$$x = 397, \ y = 593, \ z = -707, \ p = -833, \ q = -553, \ r = -97.$$

For the second case, he uses $u = 4$; then $s + u = 13$, $s + 2u = 17$, $3s + 4u = 43$ and,

consequently,

$$x = 3365, \ y = 6697, \ z = -6755, \ p = -8897, \ q = 3479, \ r = 3247.$$

31 and 32. Here Euler demonstrates the reasons why the rule just described is valid.

33.   The three numbers s, t, u are only required to satisfy $s^2 = t^2 + 2u^2$, and Euler lists

the simplest such numbers.

s	3	9	17	19	27	33	33	41	43
t	1	7	1	17	23	17	31	23	7
u	2	4	12	6	10	20	8	24	30.

Fourth method

34.   Euler returns to the considerations in section 4. Here he showed that the first two

equations of the system (1)

$$y^2 + z^2 - x^2 = p^2, \qquad z^2 + x^2 - y^2 = q^2$$

are satisfied if

$$z = a^2 + b^2, \quad y^2 - x^2 = 4ab\left(a^2 - b^2\right), \quad p = a^2 + 2ab - b^2, \quad q = a^2 - 2ab - b^2.$$

He notes that these are satisfied if we take

$$z = mn\left(a^2 + b^2\right), \quad y^2 - x^2 = 4m^2n^2\left(a^2 - b^2\right)$$

and $\quad p = mn\left(a^2 + 2ab - b^2\right), \quad q = mn\left(a^2 - 2ab - b^2\right).$

Therefore it remains to fulfill the third condition $x^2 + y^2 - z^2 = r^2$.

35, 36 and 37. Euler continues to derive a final method of solving (1) described in detail in the final sections.

Summary of Euler's fourth method

38.   Arbitrarily select numbers a and b, then calculate $A = a^2 + ab, \; B = a^2 - b^2$, then

$f = A^2 - B^2$ and $g = -2AB$. From these we get $m = f + g, \; n = f - g$. Finally we have

$$x = m^2 A - n^2 B, \qquad p = mn(a^2 + 2ab - b^2),$$

$$y = m^2 A + n^2 B, \qquad q = mn(a^2 - 2ab - b^2),$$

$$z = mn(A - B), \qquad r = mn(A + B) + (m^2 - n^2)(A - B).$$

Example 1. Let $a = 1, \; b = 2$; we have $A = 3, \; B = -2$; From these we have $f = 5$,

$g = 12, \; m = 17, \; n = -7$, and finally the desired numbers are:

$x = 965, \; y = 769, \; z = -595, \; p = -119, \; q = 833, \; r = 1081.$

This solution was found in section 22.

Example 2. Let $a = 2, \; b = 1$; and get $A = 6, \; B = 1, \; f = 35, \; g = -12$. Finally we

have $m = 23, \; n = 47$, and consequently,

$x = 965, \; y = 5383, \; z = 5405, \; p = 7567, \; q = -1081, \; r = -833.$

39.   Euler notes we should not take both the numbers *a* and *b* odd. He gives a final example and remarks that "all the solutions found with this method, are essentially different from all those calculated from the preceding methods."

Example 3. Let $a = 2$, $b = 3$; we have $A = 10$, $B = -3$, $f = 91$, $g = 60$, $m = 151$, $n = 31$; from he gets

$$x = 230893, \ y = 225127, \ z = 60853, \ p = 32767, \ q = -79577, \ r = 316687.$$