An investigation of two numbers of the form $xy(x^4 - y^4)$, of which the product and the quotient will be a square*

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§. 1.

In Diophantine Analysis, many problems occur, for the resolution of which two numbers are required of the form $xy(xx - yy)$ or even $xy(xx + yy)$, of which one divided by the other produces a square. But really the development of these formulas can in no way be obtained in general, yet we must be content to have resolved certain particular cases which require even no small amount of wisdom: just as I have shown more broadly in some discussions, where I have explained this argument with all devotion. Wherefore since the proposed formula $xy(x^4 - y^4)$, is much more complicated, and indeed encompasses a pair of such formulas as if in itself, one may not unjustly doubt whether its solution surpasses analytic forces or not.

§. 2. Indeed I would have hardly dared to undertake its solution, if I had not by such good fortune happened onto a solution of a certain most difficult problem which requires a pair of numbers of this form

$$xy(x^4 - y^4),$$

whose product is a square. Hence from the solution discovered by me it was permitted to assign reciprocally such numbers which satisfied the proposed condition.

§. 3. Since therefore it is necessary that the accepted resolution of the proposed question refer to that problem, it will be not at all beside the point to recall that problem briefly here; for although I have already discussed that problem in Volume

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XV of the new Commentaries,[a] here I am about to discuss a much easier and more elegant solution. The problem however was expressed thusly:

To find two numbers, of which the product either increased or decreased by the sum or difference of those numbers produces square numbers.

§. 4. Let the pair of sought numbers, since they cannot be integers, be set as $\frac{x}{z}$ and $\frac{y}{z}$, and moreover it is necessary that these formulas:

$$xy \pm z(x+y) \quad \text{and} \quad xy \pm z(x-y)$$

become squares. For the first of these formulas let us put $xy = aa+bb$, and it will be satisfied by assuming $z(x+y) = 2ab$. In a similar way, if for the latter we put $xy = cc+dd$, it will be necessary that $z(x-y) = 2cd$. Therefore it must be accomplished, that both values assumed for xy are rendered equal between themselves, or so that it becomes $aa+bb = cc+dd$. Then, when from the former $x+y = \frac{2ab}{z}$, from the latter indeed $x-y = \frac{2cd}{z}$, hence it is gathered that $x = \frac{ab+cd}{z}$ and $y = \frac{ab-cd}{z}$ of which therefore the product $\frac{aabb-ccdd}{zz}$ ought to be equal to $aa+bb$ and also $cc+dd$, from which it will necessarily become $zz = \frac{aabb-ccdd}{aa+bb} = \frac{aabb-ccdd}{cc+dd}$. Since therefore xy must be resolvable into a sum of two squares in a two-fold way, let us set $xy = (pp+qq)(rr+ss)$ and hence for the former formula $aa+bb$ it is taken $a = pr+qs$ and $b = ps-qr$; for the latter certainly $c = pr-qs$, then indeed $d = ps+qr$. From here therefore it will be[b]

$$ab+cd = 2rs(pp-qq) \quad \text{and} \quad ab-cd = 2pq(rr-ss)$$

whence it will produce $zz = \frac{4pqrs(pp-qq)(rr-ss)}{(pp+qq)(rr+ss)}$.

§. 5. Since therefore this fraction must be a square, also the product of the numerator into the denominator, which is

$$4pqrs(p^4-q^4)(r^4-s^4),$$

will have to be a square, which is evidently reduced to this product

$$pq(p^4-q^4) \times rs(r^4-s^4),$$

[a] *Solutio problematis quo duo quaeruntur numeri, quorum productum tam summa, quam differentia eorum, sive auctum sive minutum fiat quadratum*, *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, v.XV (1770), 1771, p.29–50. E405 in the Eneström index.

[b] There is a sign error in what follows. It should read $ab-cd = 2pq(ss-rr)$. The ability to remove negative signs and other such freedoms are addressed in §22.

or even this fraction $\frac{pq(p^4-q^4)}{rs(r^4-s^4)}$ must be reduced to a square, which therefore is the very same question that I have undertaken to clarify here.

§. 6. However, once that problem had been proposed to me, after many efforts were made in vain at last for the pair of sought numbers $\frac{x}{z}$ and $\frac{y}{z}$ I elicited these values $\frac{3\ .\ 29^2}{8\ .\ 9^2}$ and $\frac{5\ .\ 29^2}{32\ .\ 11^2}$, from which case for the letters p, q, r, s I in turn deduced those values:

$$p = 12,\ \ q = 1,\ \ r = 16 \text{ and } s = 11,$$

let us see how they satisfy our question. It will therefore be

$$\begin{array}{l|l} p = 12 & r = 16 \\ q = 1 & s = 11 \\ p+q = 13 & r+s = 27 \\ p-q = 11 & r-s = 5 \\ pp+qq = 5\ .\ 29 & rr+ss = 13\ .\ 29. \end{array}$$

From here further on we gather it to be:

$$\begin{aligned} pq(p^4-q^4) &= 4\ .\ 3\ .\ 13\ .\ 11\ .\ 5\ .\ 29 \\ rs(r^4-s^4) &= 16\ .\ 11\ .\ 3\ .\ 9\ .\ 5\ .\ 13\ .\ 29 \end{aligned}$$

from which is concluded $\frac{pq(p^4-q^4)}{rs(r^4-s^4)} = \frac{1}{4\cdot 9}$.

§. 7. Since therefore the case which satisfies the question here proposed is evident to us, its consideration will be able to lead us through to investigating other solutions. Therefore let us observe certain notable relations occurring among the values found, where at once this notable harmony is discovered, that the formulas $pp+qq$ and $rr+ss$ have a common factor 29, while the other factors are 5 and 13, all certainly the sums of two squares, just as the nature of the thing requires.

§8. Beginning from this condition[c] therefore let us set

$$pp+qq = (aa+bb)(xx+yy) \text{ and } rr+ss = (cc+dd)(xx+yy)$$

so that $xx+yy$ is a factor common to either formula, and from here we will obtain the following values:

$$\begin{aligned} p &= ax+by; \quad r = cx+dy \\ q &= bx-ay; \quad s = dx-cy \end{aligned}$$

and therefore

$$\begin{aligned} p+q &= (a+b)x+(b-a)y; \quad r+s = (c+d)x+(d-c)y \\ p-q &= (a-b)x+(b+a)y; \quad r-s = (c-d)x+(d+c)y. \end{aligned}$$

[c] *Ab hac igitur conditioni.* I'm reading it as *conditione.*

§. 9. However, further on let us bring about, so that two terms one from each side may so far mutually cancel each other out,[d] and considering the example just given, we find that the formula $p-q=11$ is equal to the formula $s=11$, from which let us establish in general this equality $p-q=s$, and from here arises this equation: $(a-b)x+(b+a)y=dx-cy$, from which a ratio between x and y is defined voluntarily; for it becomes $\dfrac{x}{y}=\dfrac{a+b+c}{d+b-a}$. Wherefore in general let us set $x=a+b+c$ and $y=d+b-a$. Although however the question seems restricted by this method, nevertheless after the matter has been carefully weighed clearly no restriction has been made. For since on both sides whatever multiples of the letters p and q and also r and s in the same way satisfy, it will always be permitted to take such multiples that make $p-q=s$.

§10. In addition it will even help to have observed that the formula $p+q$ in the example is equal to $cc+dd$ itself; for which reason in general let us put $p+q=cc+dd$, by which position however a huge restriction is certainly introduced. Therefore after the values just found have been substituted in place of x and y, the following equation is obtained:

$$(a+b)(a+b+c)+(b-a)(b-a+d)=cc+dd \quad \text{or}$$
$$(a+b)^2+c(a+b)+(b-a)^2+d(b-a)=cc+dd$$

to which equation if $\frac{1}{4}(cc+dd)$ is added to both sides, this will come out:

$$(a+b+\tfrac{1}{2}c)^2+(b-a+\tfrac{1}{2}d)^2=\tfrac{5}{4}(cc+dd)$$

where in the left part is contained the sum of two squares. It is certainly evident that the right member contains the sum of two squares in a two-fold way either $(c+\frac{1}{2}d)^2+(d-\frac{1}{2}c)^2$ or even $(c-\frac{1}{2}d)^2+(d+\frac{1}{2}c)^2$. Hence, just as we may set for both parts whichever square equal to either one or the other, the following four combinations occur here:

$$\begin{array}{cc}
\text{I.} & \text{II.} \\
a+b+\tfrac{1}{2}c=c+\tfrac{1}{2}d, & a+b+\tfrac{1}{2}c=d-\tfrac{1}{2}c \\
b-a+\tfrac{1}{2}d=d-\tfrac{1}{2}c, & b-a+\tfrac{1}{2}d=c+\tfrac{1}{2}d, \\
 & \\
\text{III.} & \text{IV.} \\
a+b+\tfrac{1}{2}c=c-\tfrac{1}{2}d, & a+b+\tfrac{1}{2}c=d+\tfrac{1}{2}c \\
b-a+\tfrac{1}{2}d=d+\tfrac{1}{2}c, & b-a+\tfrac{1}{2}d=c-\tfrac{1}{2}d,
\end{array}$$

§. 11. So from these four cases it is appropriate that the one be chosen which agrees with the example. But if we compare the formulas found to this point with

[d] *ut utrinque duo tantum termini se mutuo destruant...* I think he means that after the choices in this section, $p-q$ will cancel s in the fraction $\frac{pq(p^4-q^4)}{rs(r^4-s^4)}$.

the example we will discover $a = 2, b = 1, c = 2, d = 3$, from which it becomes $x = 5, y = 2$, so that it is

$$xx + yy = 29;\ aa + bb = 5;\ cc + dd = 13;$$

and from here all the remaining values agree with the example completely. Since therefore it is $a + b + \frac{1}{2}c = 4$ and $b - a + \frac{1}{2}d = \frac{1}{2}$, the combination is chosen which puts forth these same values, whereas it will easily arise that the fourth must be used. We will therefore have $a + b = d$, $b - a = c - d$, and from here the letters c and d are defined through a and b so that $c = 2b$ and $d = a + b$ from which a little later it will be $x = a + 3b; y = 2b$. Therefore when these values have been set up single factors of each formula

$$pq(p + q)(p - q)(pp + qq) \text{ and } rs(r + s)(r - s)(rr + ss)$$

are found to be expressed in the following way[e]

$$\begin{array}{l}
p = aa + 3ab + 2bb = (a + b)(a + 2b) \\
q = 3bb - ab = b(3b - a) \\
p + q = aa + 2ab + 5bb \\
p - q = aa + 4ab - bb \\
pp + qq = (aa + bb)(xx + yy) \\
\hline
r = 4ab + 8bb = 4b(a + 2b) \\
s = aa + 4ab - bb \\
r + s = aa + 8ab + 7bb = (a + b)(a + 7b) \\
r - s = 9bb - aa = (3b + a)(3b - a) \\
rr + ss = (aa + 2ab + 5bb)(xx + yy).
\end{array}$$

§. 12. Let us now join together the individual factors of each formula, and we will find:

$$\begin{aligned}
pq(p^4 - q^4) &= (a + b)(a + 2b)b(3b - a)(aa + 2ab + 5bb)\ \times \\
&\quad (aa + 4ab - bb)(aa + bb)(xx + yy) \\
rs(r^4 - s^4) &= 4b(a + 2b)(aa + 4ab - bb)(a + b)(a + 7b) \\
&\quad (3b + a)(3b - a)(aa + 2ab + 5bb)(xx + yy).
\end{aligned}$$

But if we divide the former by the latter it will be

$$\frac{pq(p^4 - q^4)}{rs(r^4 - s^4)} = \frac{aa + bb}{4(a + 7b)(3b + a)}.$$

Wherefore, so that this fraction is equal to a square, such values are required for the letters a and b, so that this fraction $\dfrac{aa + bb}{(a + 7b)(3b + a)}$ becomes a square, or even its

[e]The final equation involving $rr + ss$ was mistakenly written $rr - ss$ in the original.

inverse $\dfrac{(a+3b)(a+7b)}{aa+bb}$, because it indeed happens certainly for our example, by taking $a=2$ and $b=1$, for then the value of this latter fraction will be 9. Therefore all work returns to this point, that this fraction is reduced to a square.

§. 13. Let us put here $\dfrac{a}{b}=t$, so that the formula that must equal a square is $\dfrac{(t+3)(t+7)}{tt+1}$ and I am about to surrender here an utterly singular method of discovering from whatever known value you wish countless others. To this end, it will help to have recorded many cases which offer themselves as if at will which are $t=2, t=1, t=-2, t=\infty, t=-3, t=-7$. Let us here show in what way therefore from these known values other new ones can draw forth. Before all other things however the product from the numerator into the denominator must be considered, which is:

$$t^4+10t^3+22tt+10t+21$$

which therefore must necessarily be reduced to a square.

§. 14. Since this first term is so much a square, let us thus find its root, so that even the second term is removed; wherefore let this formula be set equal to this square:

$$(tt+5t+v)^2,$$

and hence the following equation arises:

$$22tt+10t+21=(2v+25)tt+10tv+vv,$$

which is reduced to this:

$$-3tt+10t+21=2vtt+10tv+vv,$$

and this equation contains the two letters t and v, either one of which rises to two dimensions, and therefore, as long as either is seen as known the other will receive twin values, which if they are indicated by t and t', and also by v and v' from the nature of the equations it is evident that it will be:

$$t+t'=\frac{10(1-v)}{2v+3}, \quad \text{then indeed} \quad v+v'=-2t(t+5),$$

with the help of which formulas as soon as the values for t and v become clear thence new ones for the same letters will be dug up and from these in the same way new ones once more, so that such operations can be continued without end.

§. 15. Therefore when some values for t are already known, we may see which values of v correspond to them, the determination of which can be attacked from the last equation. Thus if it is $t=2$, this equation turns out to be $vv+28v=29$, from which arise these two values $v=1$ and $v=-29$. On behalf of the second value $t=1$ arises $v=2$ and $v=-14$. On behalf of the third value $t=-2$ produces $v=1$

and $v = 11$. On behalf of the fourth $t = \infty$ both terms containing the square tt must cancel themselves out, and thus it will be $2v = -3$ or $v = -\frac{3}{2}$, then indeed to this value $v = -\frac{3}{2}$ corresponds to value $t = -\frac{3}{4}$. On behalf of the fifth value $t = -3$ we obtain the value $v = 6$. And finally the case $t = -7$ offers $v = -14$.

§. 16. But if a pair of values for the letters t and v are already accepted in this way, new ones will be formed from these with the help of these formulas:

$$t' = \frac{10(1-v)}{2v+3} - t; \quad v' = -2t(t+5) - v.$$

Therefore let us begin with the case $t = 2$ and $v = 1$ and the whole operation proceeds in the following way:

$$\begin{array}{rrrrrr} t = 2, & -2, & -2, & 2, & \frac{-82}{11}, & \frac{262}{649}, \\ v = 1, & 11, & 1, & -29, & \frac{-919}{121}. & \end{array}$$

In this operation are already contained the cases known at the beginning whence we can overlook them now.

§. 17. Let us proceed therefore to the fourth case in which $v = -\frac{3}{2}$ and $t = -\frac{3}{4}$, where we may write the values of v and t[f] in the inverse way, since previously t' came out $= \infty$. Therefore the operation will hold itself thus:[g]

$$\begin{array}{rrrr} v = -\frac{3}{2}, & \frac{63}{8}, & \frac{77}{18}, & -\frac{3\,.\,67\,.\,137}{32\,.\,169}, \\ t = -\frac{3}{4}, & -\frac{35}{12}, & \frac{25}{312}. & \end{array}$$

§. 18. Let us now assume $t = -3$ and $v = 6$, and by setting up the operation in the same way these values arise:

$$\begin{array}{rrrr} t = -3, & -\frac{1}{3}, & -\frac{41}{3}, & \frac{267}{31}, \\ v = 6, & -\frac{26}{9}, & -9\,.\,26. & \end{array}$$

The initial terms can even be inverted in this way:

$$\begin{array}{rrrr} v = 6, & 6, & -\frac{26}{9}, & -9\,.\,26, \\ t = -3, & -\frac{1}{3}, & -\frac{41}{3}. & \end{array}$$

It is evident however that no new values emerge since all are already contained in the preceding series.

§. 19. Finally let us unwind the last case in which $t = -7$ and $v = -14$ for which the following values are dug up:[h]

$$\begin{array}{rrrr} t = -7, & 1, & -\frac{17}{7}, & -\frac{503}{7\,.\,47}, \\ v = -14, & 2, & \frac{514}{49}. & \end{array}$$

[f] The original has z instead of t.

[g] The final value of v was incorrectly given as $-\frac{164\,.\,27}{11\,.\,11}$ in the original. I wrote out the prime power factors of the corrected answer, $-\frac{27537}{5408}$.

[h] The original was missing a negative sign in front of $\frac{503}{7\,.\,47}$, a 1 from the 514 numerator, and the fraction bar from $\frac{514}{49}$.

However with the first terms v and t having been placed in the inverse way it becomes:[i]

$$\begin{array}{rrrr} v=-14, & -14, & 2, & \frac{514}{49}, \\ t=-7, & 1, & -\frac{17}{7}, & -\frac{503}{329}, \end{array}$$

which numbers however already occur in the preceding series. The rest must be noted, that the second case, in which $t=1$ and $v=2$ or $=-14$ is repeated again here, which was the case overlooked above.

§. 20. Therefore through these operations suitable values for t found in the following way will hold themselves:[j]

$$\begin{array}{l} \text{I. } t=2;\ \text{II. } t=-2;\ \text{III. } t=-\frac{82}{11};\ \text{IV. } t=\frac{262}{649}; \\ \text{V. } t=-\frac{3}{4};\ \text{VI. } t=-\frac{35}{12};\ \text{VII. } t=\frac{25}{312};\ \text{VIII. } t=-3; \\ \text{IX. } t=-\frac{1}{3};\ \text{X. } t=-\frac{41}{3};\ \text{XI. } t=\frac{267}{31};\ \text{XII. } t=-7; \\ \text{XIII. } t=1;\ \text{XIV. } t=-\frac{17}{7};\ \text{XV. } t=-\frac{503}{329}. \end{array}$$

Therefore all of these values one at a time supply a solution of the proposed question; just as we will show in the following problem.

From whichever suitable value found for t *to assign four numbers* p, q, r, s, *so that the product or the quotient of these formulas* pq(p⁴−q⁴) *and* rs(r⁴−s⁴) *become square.*

Solution.

§. 21. Since it [i.e., t] is $\frac{a}{b}$, and both numbers a and b will be considered likewise in the integers, from which the letters p, q, r, s will be determined when derived in the following way:[k]

$$\begin{array}{l|l} p=(a+b)(a+2b) & r=4b(a+2b) \\ q=b(3b-a) & s=aa+4ab-bb \\ p+q=aa+2ab+5bb & r+s=(a+b)(a+7b) \\ p-q=aa+4ab-bb & r-s=(3b+a)(3b-a) \\ pp+qq=(aa+bb)(x^2+y^2) & rr+ss=(aa+2ab+5bb)(x^2+y^2) \\ =(aa+bb)(aa+6ab+13bb) & =(aa+2ab+5bb)(aa+6ab+13bb). \end{array}$$

§. 22. About these formulas it must be observed 1) if any one of the numbers p, q, r, s, would come out as a negative, that in its place always the positive can be written; 2) if it would come out either $q>p$ or $s>r$ that those values can always be permuted among themselves, such that the letter p may indicate the greater number, q indeed the smaller, and in a similar way r the greater and s the smaller;

[i] The original was missing the negative sign on $-\frac{503}{329}$.

[j] The original was missing a fraction bar in $\frac{262}{649}$ and a minus sign in $-\frac{503}{329}$.

[k] In the original, $r-s$ was incorrectly listed as $(3b+a)(8b-a)$.

3) if it happens that the numbers p and q have a common divisor, that it is always permitted to remove it through division, which must likewise be held for the letters r and s. 4) It is also evident, that so much in place of the pair of letters p and q as r and s their sum and difference can be written: For if we put

$P = p + q, Q = p - q, R = r + s, S = r - s$

it will become $PQ(P^4 - Q^4) = 8pq(p^4 - q^4)$ and in a similar way

it will become $RS(R^4 - S^4) = 8rs(r^4 - s^4)$,

and thus also either the quotient or the product of these new formulas will also be a square. 5) This transformation offers a notable use if the letters p, q, r, s were odd; for then the capital letters P, Q, R, S can be reduced, and thus it will arrive at smaller numbers: for if we put

$$P = \frac{p+q}{2}, Q = \frac{p-q}{2}, R = \frac{r+s}{2}, S = \frac{r-s}{2}$$

it will become $PQ(P^4 - Q^4) = \dfrac{pq(p^4 - q^4)}{8}$ and $RS(R^4 - S^4) = \dfrac{rs(r^4 - s^4)}{8}$. Therefore following this precept for the found values of t itself let us assign the letters p, q, r, s in the following examples.

Example 1,
where $t = 2$.

§. 23. Thus here it will be $a = 2$ and $b = 1$ and hence it will become

$$p = 3 \,.\, 4, q = 1, r = 4 \,.\, 4, s = 11,$$

which therefore with their derivates are arranged thus:

$p = 4 \,.\, 3$	$r = 4 \,.\, 4$
$q = 1$	$s = 11$
$p + q = 13$	$r + s = 3 \,.\, 9$
$p - q = 11$	$r - s = 5$
$pp + qq = 5 \,.\, 29$	$rr + ss = 13 \,.\, 29.$

Again, let us omit here the cases $t = -2$ and $t = 1$, $t = -3$, $t = -7$, $t = -\frac{1}{3}$, because they provided unsuitable [i.e., trivial] solutions.

Example 2,
where $t = -\frac{3}{4}$.

§. 24. Since therefore it is $a = -3$ and $b = 4$, it will become in this case

$p = 1$	$r = 4 \,.\, 4$
$q = 4 \,.\, 3$	$s = 11$
$p + q = 13$	$r + s = 27$
$p - q = 11$	$r - s = 5$
$pp + qq = 5 \,.\, 29$	$rr + ss = 13 \,.\, 29,$

which values however differ with the preceding only in this, that p and q have been permuted; thus from here no new solution emerges.

Example 3,
where $t = -\frac{17}{7}$.

§. 25. Here therefore it must be assumed $a = -17$, $b = 7$, whence these values arise, having been depressed by 2 and 4 of course:

$$\begin{array}{l|l} p = 3\,.\,5 & r = 3\,.\,7 \\ q = 7\,.\,19 & s = 59. \end{array}$$

Now since all these numbers are odd, in their place let semi-sums and semi-differences be written and therefore this new solution of the problem arises:

$$\begin{array}{l|l} p = 2\,.\,37 & r = 40 \\ q = 59 & s = 19 \\ p + q = 133 & r + s = 59 \\ p - q = 15 & r - s = 21 \\ pp + qq = 53\,.\,169 & rr + ss = 53\,.\,37. \end{array}$$

Where all the non square factors on both sides cancel themselves out.

Example 4,
where $t = -\frac{41}{3}$.

§. 26. Here, having assumed $a = -41$ and $b = 3$ the values p, q, r, s having been depressed as much as is permitted will be:

$$p = 7\,.\,19\ ;\ q = 3\,.\,5\ ;\ r = 3\,.\,7\ ;\ s = 59,$$

which case agrees perfectly with the preceding.

Example 5,
where $t = -\frac{35}{12}$.

§. 27. Since therefore it ought to be taken $a = -35$ and $b = 12$ values for p, q, r, s from here will be:[1]

$$\begin{array}{l|l} p = 12\,.\,71 = 852 & r = 599 \\ q = 11\,.\,23 = 253 & s = 11\,.\,48 = 528 \\ p + q = 5\,.\,13\,.\,17 & r + s = 23\,.\,49 \\ p - q = 599 & r - s = 71 \\ pp + qq = 37^2\,.\,577 & rr + ss = 5\,.\,13\,.\,17\,.\,577, \end{array}$$

[1] Euler enacts some of the permutations he outlined above, namely p and q are swapped, and r and s are swapped and made positive. Also, $p + q = 1105 = 5\,.\,13\,.\,17$ was mistakenly given as $5\,.\,21$ [i.e., 105] in the original.

where again all non square factors occur on both sides. Thence it is clear that the same solution results farther on from the case $t = -\frac{82}{11}$, since

$$82^2 + 11^2 = 5(35^2 + 12^2).$$

§. 28. Thus besides the case known long ago, where

$$p = 12,\ q = 1,\ r = 16,\ s = 11,$$

which has served us as a semblance of a pattern in this investigation we have gained two other new solutions, which correspond to not exceedingly large numbers. Indeed the remaining four cases found for t:

$$\frac{25}{312},\ \frac{267}{31},\ \frac{503}{329},\ \frac{262}{649}$$

led to numbers not exceedingly large, which it is not worthwhile to explain. Still in these operations occur many tricks of calculation hardly known until now, the Analysis of which should be judged no meager development to undertake.

§. 29. From here now the Problem treated in Volume XV of the New Commentaries[m] can be resolved much more suitably and neatly and can be obtained through absolute numbers, which solution I append here.

Problem.

To find two numbers, of which the product either increased or decreased by the sum or difference of those numbers produces square numbers.

Solution.

§. 30. Having put the desired numbers as $\frac{x}{z}$ and $\frac{y}{z}$ we have already seen above that $x = \frac{ab+cd}{z}$ and $y = \frac{ab-cd}{z}$; from there after the letters p, q, r, s, were introduced it was

$$ab + cd = 2rs(pp - qq) \text{ and } ab - cd = 2pq(rr - ss).$$

On account of which the sought numbers will be:

$$\frac{x}{z} = \frac{2rs(pp-qq)}{zz} \text{ and } \frac{y}{z} = \frac{2pq(rr-ss)}{zz}.$$

Further on we found moreover that $zz = \frac{4pqrs(pp-qq)(rr-ss)}{(pp+qq)(rr+ss)}$ with the value of which having been substituted, both sought numbers will be:

$$\frac{x}{z} = \frac{(pp+qq)(rr+ss)}{2pq(rr-ss)} \text{ and } \frac{y}{z} = \frac{(pp+qq)(rr+ss)}{2rs(pp-qq)}.$$

[m] *op. cit.*

§. 31. Therefore since we gave in the examples above three solutions in absolute numbers, if from these we bring out values for the letters p, q, r, s, we will obtain the following three numerical solutions.

I. Solution,
obtained from §. 23.

$$\frac{x}{z} = \frac{5\,.\,29\,.\,13\,.\,29}{2\,.\,12\,.\,3\,.\,9\,.\,5} = \frac{13\,.\,29^2}{8\,.\,9^2}$$
$$\frac{y}{z} = \frac{5\,.\,29\,.\,13\,.\,29}{13\,.\,11\,.\,32\,.\,11} = \frac{5\,.\,29^2}{32\,.\,11^2},$$

which is the solution found by me in the first place.[n]

II. Solution,
obtained from §. 25.

$$\frac{x}{z} = \frac{53\,.\,169\,.\,53\,.\,37}{4\,.\,37\,.\,59\,.\,59\,.\,21} = \frac{13^2\,.\,53^2}{3\,.\,4\,.\,7\,.\,59^2}$$
$$\frac{y}{z} = \frac{53\,.\,169\,.\,53\,.\,37}{38\,.\,40\,.\,133\,.\,15} = \frac{37\,.\,13^2\,.\,53^2}{3\,.\,7\,.\,4^2\,.\,5^2\,.\,19^2}.$$

III. Solution,
obtained from §. 27.[o]

$$\frac{x}{z} = \frac{37^2\,.\,577\,.\,5\,.\,13\,.\,17\,.\,577}{24\,.\,71\,.\,11\,.\,23\,.\,23\,.\,49\,.\,71} = \frac{5\,.\,13\,.\,17\,.\,37^2\,.\,577^2}{11\,.\,24\,.\,7^2\,.\,23^2\,.\,71^2}$$
$$\frac{y}{z} = \frac{37^2\,.\,577\,.\,5\,.\,13\,.\,17\,.\,577}{2\,.\,528\,.\,599\,.\,5\,.\,13\,.\,17\,.\,599} = \frac{37^2\,.\,577^2}{2\,.\,528\,.\,599^2}.$$

§. 32. For the sake of curiosity let me append here the solution satisfied by the largest numbers thus far, which the case found above in §. 20. makes available,

[n]Just for completion, getting a common denominator makes $z = 313632$, $x = 5291572$ and $y = 340605$. The four expressions $(\frac{x}{z})(\frac{y}{z}) \pm (\frac{x}{z} \pm \frac{y}{z})$ are the squares of: $\frac{1889118}{313632} = \frac{9541}{1584}$, $\frac{189486}{313632} = \frac{29}{48}$, $\frac{1831698}{313632} = \frac{841}{144}$, and $\frac{499554}{313632} = \frac{841}{528}$.

[o]The original propagates the error from §27, listing $5\,.\,21$ instead of $5\,.\,13\,.\,17$ for the $p+q$ factor. Also, it's not clear why 528 is left unfactored.

namely $t = \dfrac{25}{312}$, whence it becomes $a = 25$ and $b = 312$. From here moreover the following values are deduced:

$$\begin{array}{l|l}
p = 3\,.\,8\,.\,13\,.\,911 & r = 3\,.\,11\,.\,13\,.\,32\,.\,59 \\
q = 11\,.\,59\,.\,337 & s = 65519 \\
p + q = 5\,.\,17\,.\,61\,.\,97 & r + s = 31^2\,.\,911 \\
p - q = 65519 & r - s = 337\,.\,47^2 \\
pp + qq = 313^2\,.\,1312897 & rr + ss = 5\,.\,17\,.\,61\,.\,97\,.\,1312897,
\end{array}$$

from which the sought numbers will be:

$$\begin{aligned}
\frac{x}{z} &= \frac{5\,.\,17\,.\,61\,.\,97\,.\,313^2\,.\,1312897^2}{3\,.\,11\,.\,13\,.\,59\,.\,4^2\,.\,31^2\,.\,47^2\,.\,337^2\,.\,911^2} \\
\frac{y}{z} &= \frac{313^2\,.\,1312897^2}{3\,.\,11\,.\,13\,.\,8^2\,.\,59\,.\,65519^2}.
\end{aligned}$$