A solution of a most difficult problem, in which the two formulas: $aaxx + bbyy$ & $aayy + bbxx$ must be rendered as squares*

Leonhard Euler

§. 1.

I do not hesitate to pronounce this question not only very difficult to solve but also of greatest importance in Analysis. For first I labored long enough in vain in unraveling this; then indeed the solution that I at last obtained demands many notable tricks of calculation which seem to bring in considerable[a] developments in universal Diophantine Analysis. As however this question concerns two pairs of squares aa, bb and xx, yy, neither of them can be taken at will, but both require equal diligence and shrewdness.

§. 2. Let us therefore put

$$aaxx + bbyy = zz \text{ and } aayy + bbxx = vv,$$

and with these formulas being added and subtracted it comes out

$$(aa + bb)(xx + yy) = zz + vv \quad \text{and}$$

$$(aa - bb)(xx - yy) = zz - vv,$$

indeed from which I first hoped to be able to derive a solution; because the sum of the squares $zz + vv$ is required to be resolvable into two squares in many ways: for then it is indeed plain, that the formula $zz - vv$ must involve many factors. Meanwhile however this consideration seems to convey hardly anything towards finding a solution. For from there, after much work I was finally barely able to draw out a single solution, by which I found $a = 5, b = 3, x = 7, y = 4$. From here it now becomes

$$aaxx + bbyy = 35^2 + 12^2 = 37^2$$

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[a] *haud contemnenda*

$$aayy + bbxx = 20^2 + 21^2 = 29^2.$$

In truth, to explain my useless attempts more broadly attains utterly nothing because at last I arrived at a general and sufficiently elegant solution.

§. 3. First therefore so that the formula $aaxx + bbyy$ renders as a square I put $\frac{ax}{by} = \frac{pp-qq}{2pq}$; meanwhile for the other formula[b] I put $\frac{ay}{bx} = \frac{rr-ss}{2rs}$, of which the former divided by the latter puts forth $\frac{xx}{yy} = \frac{rs(pp-qq)}{pq(rr-ss)}$, where if it is multiplied on both sides by $\frac{ppqq}{rrss}$ there will arise $\frac{ppqqxx}{rrssyy} = \frac{pq(pp-qq)}{rs(rr-ss)}$, and thus we have induced a complete resolution to these double formulas totally similar to each other $pq(pp-qq)$ and $rs(rr-ss)$, of which the one divided by the other should produce a square, or what amounts to the same thing[c], that their product should come out a square, in regard to which task many Geometers have concerned themselves, but no general enough solution has yet been found by anyone, for which reason I have found not only many particular solutions toward this intention that are suitable enough, but at last I happened to stumble upon a general solution, by which the territory of diophantine Analysis will be advanced greatly.

§. 4. However, but if we will have by chance come upon [something] of the sort where

$$\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{tt}{uu},$$

from there at once we deduce $\frac{pqx}{rsy} = \frac{t}{u}$, and thus $\frac{x}{y} = \frac{rst}{pqu}$, which after the fraction has been reduced to lowest terms, it is put $x = rst$ and $y = pqu$. From here, since we had $\frac{ax}{by} = \frac{pp-qq}{2pq}$, and thus $\frac{a}{b} = \frac{u(pp-qq)}{2rst}$, which after the fraction has been reduced to lowest terms it is taken again

$$a = u(pp-qq) \text{ and } b = 2rst.$$

§. 5. Here therefore can a table be neatly called into use which I have not given previously in this discussion, in which of this formula:

$$AB(AA-BB),$$

I have shown the non-square factors. Because if indeed from there two cases containing the same non-square factors are brought out, their product will certainly be a square, and therefore will supply the solution of our Problem. However in the table such values immediately offer themselves: $p = 5, q = 2, r = 6, s = 1$. For hence will be $\frac{pq(pp-qq)}{rs(rr-ss)} = 1$, and thus $t = 1$ and $u = 1$ from which therefore we will have $\frac{x}{y} = \frac{rs}{pq} = \frac{3}{5}$ and $\frac{a}{b} = \frac{pp-qq}{2rs} = \frac{7}{4}$. Therefore from here we gather $a = 7, b = 4, x = 5, y = 3$ since it is permitted to permute the letters a and b as well as x and y among themselves, and thus this case agrees with what was related before.

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[b]The original mistakenly has ax/by in the next formula.

[c]*vel quod eodem redit*

§. 6. By a similar method the alleged table gives these values in addition: $p = 5, q = 2, r = 8, s = 7$, from which it becomes $\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{1}{4}$, therefore $t = 1$ and $u = 2$. Therefore from here we will have

$$\frac{x}{y} = \frac{rs}{2pq} = \frac{14}{5} \text{ and } \frac{a}{b} = \frac{pp-qq}{rs} = \frac{3}{8}.$$

For which reason it can be assumed $a = 8, b = 3, x = 14, y = 5$ from which it becomes $aaxx + bbyy = 113^2$ and $aayy + bbxx = 58^2$, which solution[d] disagrees very little with the preceding.

§. 7. To this point another case can be drawn out from the table, where $p = 6, q = 5, r = 8, s = 3$, which gives $\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{1}{4}$ thus again $t = 1$ and $u = 2$, from which is gathered

$$\frac{x}{y} = \frac{rs}{2pq} = \frac{2}{5} \text{ and } \frac{a}{b} = \frac{pp-qq}{rs} = \frac{11}{24}.$$

Therefore having supposed $a = 24, b = 11, x = 5, y = 2$, it will become

$$aaxx + bbyy = 122^2 \text{ and } aayy + bbxx = 73^2.$$

§. 8. However, since only individual solutions so far are discovered in this way, and the table is restricted to such narrow limits[e], here we are especially going to investigate into more general formulas, which may contain at the same time an infinite multitude of solutions, since I have observed that it can be done in many ways, even if these formulas may only exhibit particular solutions so far. Therefore, let us convey several particular solutions of this type [for the common good], from which it would be permitted to derive innumerable other solutions, which after having been explained, we will approach at last a general solution.

First particular solution.

§. 9. Let us assume at once that $s = q$ and $r = p + q$, by which agreement our general fraction $\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{tt}{uu}$ is reduced to this simple form $\frac{p-q}{p+2q} = \frac{tt}{uu}$ from which we deduce $\frac{p}{q} = \frac{uu+2tt}{uu-tt}$. On account of which if we assume $p = uu + 2tt$ and $q = uu - tt$ it will become $r = 2uu + tt$ and $s = uu - tt$. From these values therefore is gathered

$$\frac{x}{y} = \frac{t(2uu+tt)}{u(uu+2tt)} \text{ and } \frac{a}{b} = \frac{3tu}{2(uu-tt)}$$

and thus

$$a = 3tu, b = 2(uu-tt), x = t(2uu+tt), y = u(uu+2tt).$$

Then from these values it will be

$$ax = 3ttu(2uu+tt) \text{ and } by = 2u(uu-tt)(uu+2tt).$$

[d]It seems like Euler is permuting choices so that $a > b$ and $x > y$ throughout. Also, $aayy + bbxx$ is given incorrectly here as 38^2 in the original.

[e]The text has *imites* rather than *limites*.

From here we thus deduce[f]

$$z = u((uu-tt)^2 + (uu+2tt)^2) = u(2u^4 + 2ttuu + 5t^4).$$

In a similar way when it is

$$ay = 3tuu(uu+2tt) \text{ and } bx = 2t(uu-tt)(2uu+tt),$$

whence we deduce

$$v = t((uu-tt)^2 + (2uu+tt)^2) = t(2t^4 + 2ttuu + 5u^4).$$

§. 10. From here therefore many particular solutions can be deduced with little effort, since it is permitted to assume whatever numbers you wish for the letters t and u, not only in small numbers but also it will be permitted to assume values as large as you like, in a way they could not be obtained at all by means of the table previously used. Therefore it will be worthwhile to illustrate these formulas through examples, certainly as long as we assign values to the letters t and u arbitrarily. But because the letters t and u are permuted between themselves, let us assign to u itself the larger values, indeed t the smaller, because the case $t = u$ would give nothing. From here, let us place many examples together into the following table in full view:

u	2	3	3	4	4	5	5	5	5	6	6
t	1	1	2	1	3	1	2	3	4	1	5
a	1	9	9	2	18	5	5	45	10	9	45
b	1	16	5	5	7	16	7	32	3	35	11
x	3	19	44	11	123	17	36	177	88	73	485
y	4	33	51	24	136	45	55	215	95	228	516
z	5	555	471	122	2410	725	425	10525	925	8007	22551
v	5	425	509	73	2595	353	373	11211	986	3277	23825

Second particular solution

§. 11. Let it remain $r = p + q$, and let it be assumed $s = p$ and it will be

$$\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{tt}{uu} = \frac{p-q}{2p+q},$$

from which it is gathered $\frac{p}{q} = \frac{uu+tt}{uu-2tt}$. Therefore let it be assumed $p = uu + tt$ and $q = uu - 2tt$, and it will be $r = 2uu - tt$ and $s = uu + tt$. From these values it follows that it will be

$$\frac{x}{y} = \frac{rst}{pqu} = \frac{t(2uu-tt)}{u(uu-2tt)} \text{ and } \frac{a}{b} = \frac{u(pp-qq)}{2rst} = \frac{3tu}{2(uu+tt)}.$$

[f] The original mistakenly has a $-5t^4$ term instead of $+5t^4$.

On account of which the four roots of our squares will be:

$$a = 3tu; b = 2(uu + tt); x = t(2uu - tt); y = u(uu - 2tt),$$

which solution hardly differs from the preceding one, because tt here is supposed negative, with the root t still remaining, from which the following values for z and v are deduced:

$$z = u(2u^4 - 2ttuu + 5t^4) \text{ and } v = t(2t^4 - 2ttuu + 5u^4),$$

or thanks to calculation

$$z = u((uu + tt)^2 + (uu - 2tt)^2)$$

$$v = t((uu + tt)^2 + (2uu - tt)^2).$$

§. 12. Even if these formulas differ so little from the preceding, they clearly supply unlike solutions in numbers; wherefore as before let us take simpler values in place of t and u and let us represent numerical solutions in the following table, where it must be noted, if negative values come out in place of a, b, x, y, that in their place positive ones can always be written.

u	1	2	3	3	4	4	5	5	5	6
t	1	1	1	2	1	3	1	3	4	1
a	3	3	9	9	6	18	15	45	30	9
b	4	5	20	13	17	25	52	68	41	37
x	1	7	17	28	31	69	49	123	136	71
y	1	4	21	3	56	8	115	35	35	204
z	5	29	447	255	970	1258	6025	6025	4325	7575
v	5	37	389	365	625	1731	3077	8511	5674	3205

Third particular solution.

§. 13. Here let us take $s = q$ and let us put $\frac{pq(pp-qq)}{rs(rr-ss)} = 1$, and it will be $p(pp - qq) = r(rr - qq)$ from which it becomes

$$qq = \frac{r^3 - p^3}{r - p} = rr + pr + pp,$$

which formula therefore must be a square. Therefore, if it is

$$qq = \left(r + \frac{1}{2}p\right)^2 + 3\left(\frac{p}{2}\right)^2$$

let it be taken $r + \frac{1}{2}p = tt - 3uu$ and $\frac{1}{2}p = 2tu$, and it will be $q = tt + 3uu$. Thus since $p = 4tu$ it will be $r = tt - 2tu - 3uu = (t + u)(t - 3u)$. Wherefore if for the preceding formulas it is $t = 1$ and $u = 1$, which values ought not to be confused with the present ones, it will be:

$$\frac{x}{y} = \frac{rs}{pq} \text{ and } \frac{a}{b} = \frac{pp - qq}{2rs}.$$

Therefore we will have $x = (t + u)(t - 3u)$ and $y = 4tu$; then indeed

$$a = (t - u)(t + 3u) \text{ and } b = 2(tt + 3uu).$$

Therefore if it is

$$ax = (tt - uu)(tt - 9uu) \text{ and } by = 8tu(tt + 3uu)$$

let it be put $(tt - uu)(tt - 9uu) = A^2 - B^2$ and it must be

$$8tu(tt + 3uu) = 2AB,$$

so that it becomes $z = A^2 + B^2$. It will therefore be $AB = 4tu(tt+3uu)$, from which let us assume $A = tt + 3uu$ and $B = 4tu$, and it will be

$$A + B = (t + 3u)(t + u) \text{ and } A - B = (t - 3u)(t - u)$$

on account of which it will clearly be $A^2 - B^2 = (tt - uu)(tt - 9uu)$ as is required, consequently it will now be

$$z = (tt + 3uu)^2 + 16ttuu = t^4 + 22ttuu + 9u^4.$$

In a similar way[g] let it be $ay = 4tu(t - u)(t + 3u) = 4AB$ and

$$bx = 2(t + u)(t - 3u)(tt + 3uu) = 2(A^2 - B^2).$$

Indeed from here it will become $v = 2(A^2+B^2)$. Therefore let us set $A+B = tt+3uu$ and $A - B = tt - 2tu - 3uu$ from which it becomes $A = tt - tu$ and $B = 3uu + tu$, which agrees excellently with the position, consequently it will be

$$v = 2(tt(t - u)^2 + uu(t + 3u)^2).$$

Behold therefore the third particular solution to our problem

$$\begin{aligned} a &= (t - u)(t + 3u);\ b = 2(tt + 3uu) \\ x &= (t + u)(t - 3u);\ y = 4tu \\ z &= (tt + 3uu)^2 + 16ttuu \\ v &= 2tt(t - u)^2 + 2uu(t + 3u)^2. \end{aligned}$$

[g] The original mistakenly has $+4AB$ instead of $= 4AB$ in the following. It also has $(tt - 3uu)$ instead of $(tt + 3uu)$ in the following equation for bx.

Where again it must be noted if negative values come forth for these letters, that they can safely be changed into positives. Therefore let us assign simpler numerical values to both the letters t and u, from which indeed the cases $t = u$, and $t = 3u$ must be excluded, and likewise the cases where t and u are odd, and from here let us compress the derived solutions in the following table[h]

t	2	1	4	1	4	5	2	5	4
u	1	2	1	4	3	2	5	4	5
a	5	7	21	39	13	33	51	17	19
b	14	26	38	98	86	74	158	146	182
x	3	15	5	55	35	7	91	63	99
y	8	8	16	16	48	40	40	80	80
z	113	233	617	2657	4153	2969	7841	11729	14681
v	58	394	386	5426	3074	1418	14522	9298	18082

General solution.

§. 14. Since all the work has been reduced to the resolution of this equation: $\frac{pq(pp-qq)}{rs(rr-ss)} = \frac{tt}{uu}$, in place of $\frac{tt}{uu}$, let us write the letter n for the sake of brevity, so that it is $t = \sqrt{n}$ and $u = 1$, whence from the invented letters p, q, r, s the sought numbers a, b, x, y will thus be determined, so that $\frac{x}{y} = \frac{rs}{pq}\sqrt{n}$ and $\frac{a}{b} = \frac{pp-qq}{2rs\sqrt{n}}$, or, if the letters a and b may be permuted among themselves, it is possible to be put $\frac{a}{b} = \frac{2rs}{pp-qq}\sqrt{n}$, which after the fractions have been reduced to lowest terms the sought numbers a, b, x, y themselves will be had. Now let me explain[i] in what way the principal equation:

$$\frac{pq(pp-qq)}{rs(rr-ss)} = n,$$

ought to be resolved straightforwardly through this new method, from which the greatest developments will overflow into universal Diophantine Analysis, since thus far this equation has been explained generally by no one.

§. 15. Here since the sole relation between the two letters p and q and between the two r and s comes into the computation, without any restriction[j] it is permissible to assume $s = q$ so that $\frac{p(pp-qq)}{r(rr-qq)} = n$; from here it is gathered $qq = \frac{p^3-nr^3}{p-nr}$. Here now again let it be set $p = rv$ and it will become $\frac{qq}{rr} = \frac{v^3-n}{v-n}$, and thus the entire investigation reverts to this, that this formula $\frac{v^3-n}{v-n}$ is equal to a square. Thus by making use[k] of a common method, the product from the numerator into the denominator, which is $v^4 - nv^3 - nv + nn$ ought to be rendered as a square by the power of which indeed immediately some values for v could be dug up, which having been

[h] In this table, three errors have been corrected. Originally, $z(5,2)$ was listed as 2919, $v(4,1)$ as 368, and $v(1,4)$ as 6426, if $z = z(t,u)$ and $v = v(t,u)$.

[i] The text has *apperiam* rather than *aperiam*.

[j] I'm not sure why this is true.

[k] The text has *utentes*, but I'm reading *utente* in an ablative absolute.

found the formula itself ought to be transformed through new substitutions, from which once again new values could be dug up, truly soon such enormous numbers will be arrived at that only very few values of moderate magnitude could be dug up. But indeed my new method will supply to us many solutions in small enough numbers.

§. 16. I establish then $\frac{v^3-n}{v-n} = (v-z)^2$, so that $\frac{q}{r} = v - z$, while, as we saw earlier, it is $\frac{p}{r} = v$. Therefore, after the development has been done this equation comes forth:

$$(n + 2z)vv - z(2n + z)v + n(zz - 1) = 0,$$

which is quadratic with respect to v as much as to z, and thus presents two roots. But indeed the terms arranged according to z will offer this equation:

$$(n - v)zz - 2v(n - v)z - n(1 - vv) = 0.$$

Since therefore to whatever factor of v twin factors of z correspond, if these are designated via z and z' it will be from the nature of the equations [that] $z + z' = 2v$. In a similar way to whatever factor of z twin factors of v correspond, which if they are put v and v' will be $v + v' = \frac{z(z+2n)}{2z+n}$, whence, if now whatever values are held for v and z, from these new ones [are held][1] for the same letters, namely v' and z' it will be $z' = 2v - z$ and $v' = \frac{z(z+2n)}{2z+n} - v$. And in a similar way from these values again [come] new ones two at a time, and again from here others can be discovered to infinity, and it [is done] by easy work, and in this double unfolding all the power of the new solution stands, so that by this method many values may be obtained at once without any annoying transformation and so we will have learned so far the values for v and z two at a time.

§. 17. But the quadratic equation itself offers us such primitive values as if voluntarily. For having put $v = 0$ it will become $zz - 1 = 0$, from which two values arise $z = +1$ and $z = -1$. In a similar way having put $z = 0$ it becomes $vv - 1 = 0$ and thus $v = +1$ as well as $v = -1$, so that from here now let us have four cases, from which at once new values for the letters v and z may be derived. In addition indeed even a fifth case will be able to be added, arising from the position $v = \infty$; for then the coefficient of vv itself which is $n + 2z$ must be equal to nothing, whence if it is made $z = -\frac{n}{2}$ now the equation will put on this form: $3nv + nn - 4$, from which is gathered $v = \frac{4-nn}{3n}$, which is the other value of v itself, corresponding to the value $z = -\frac{n}{2}$, while the other was $v = \infty$, and from these two values $z = -\frac{n}{2}$ and $v = \frac{4-nn}{3n}$, by the power of our formulas at once many new ones can be drawn out. Therefore from here let us explain those five cases further.

Case I,
where $v = 0$ and $z = +1$.

[1] There's no verb in this clause or in some other clauses in this section in the original. I put in some reasonable options.

§. 18. From here therefore new values originating from there are discovered through alternatingly applying our formulas:

1°) $v = \frac{1+2n}{2+n}$, $z = \frac{3n}{2+n}$;

2°) $v = \frac{4(nn+n-2)}{nn+10n+16} = \frac{4(n-1)}{n+8}$; $z = \frac{5nn-16n-16}{nn+10n+16}$.

In general though it is hardly permissible to go further. In place of n now let us restore $\frac{tt}{uu}$, and when the second value is $v = \frac{1+2n}{2+n}$ and $v - z = \frac{n-1}{2+n}$, it will be $v = \frac{p}{r} = \frac{uu+2tt}{2uu+tt}$ and $v - z = \frac{q}{r} = \frac{tt-uu}{tt+2uu}$. On account of which let us assume $p = uu + 2tt, q = tt - uu, r = 2uu + tt, s = tt - uu$, which case clearly agrees with the first particular solution given above. In a similar way let us unfold the third value of v itself which was $\frac{4(n-1)}{n+8}$ to which corresponds $z = \frac{3n}{2+n}$, from where it becomes $v - z = \frac{nn-20n-8}{(2+n)(8+n)}$. Therefore from here it will be $\frac{p}{r} = \frac{4(tt-uu)}{tt+8uu}$ and $\frac{q}{r} = \frac{t^4-20ttuu-8u^4}{(2uu+tt)(8uu+tt)}$. Let it be assumed therefore $r = (2uu + tt)(8uu + tt)$ and it will be

$$p = 4(tt - uu)(tt + 2uu) \text{ and } q = s = t^4 - 20ttuu - 8u^4.$$

Here again it is thus discovered

$$\frac{x}{y} = \frac{t(8uu + tt)}{4u(tt - uu)} \text{ and } \frac{a}{b} = \frac{3tu(5t^4 - 16ttuu - 16u^4)}{2(2uu + tt)(t^4 - 20ttuu - 8u^4)}$$

from which innumerable new solutions are discovered.

Case II,
where $v = 0$ and $z = -1$.

§. 19. Here therefore values are deduced following the formulas given above: $v = \frac{1-2n}{n-2}$; $z = -\frac{3n}{n-2}$; $v = \frac{-4(n+1)}{n-8}$. Let us explain the second value of v itself namely $\frac{1-2n}{n-2}$, to which $z = -1$ corresponds; whence it becomes $v - z = \frac{n+1}{2-n}$; therefore having put $\frac{tt}{uu}$ in place of n it will become

$$\frac{p}{r} = v = \frac{uu - 2tt}{tt - 2uu} \text{ and } \frac{q}{r} = v - z = \frac{tt + uu}{2uu - tt}.$$

Thus having assumed $r = tt - 2uu$ it will be $p = uu - 2tt$ and $q = s = tt + uu$ from which it is now evident that this case agrees with the second particular solution, since it is permitted to permute the letters p and q between themselves, and therefore it is not necessary to pursue this case further.

§. 20. Let us consider thus the third value of v itself which was $\frac{-4(n+1)}{n-8}$, to which $z = \frac{-3n}{n-2}$ corresponds. These two values having been recognized we will have $\frac{p}{r} = v$ and $\frac{q}{r} = v - z$, and thus will be obtained four letters p, q, r, s, from which again the sought numbers a, b, x, y, are easily deduced by the power of the formulas given above

$$\frac{a}{b} = \frac{2rs}{pp - qq}\sqrt{n} \text{ and } \frac{x}{y} = \frac{rs}{pq}\sqrt{n}.$$

According to which let us explain a case that must be illuminated, where $n = \frac{9}{4}$ and it will be $z = -27$ and $v = \frac{52}{23}$. Hence it will be $v - z = \frac{673}{23}$. From here[m] therefore will $\frac{p}{r} = \frac{53}{23}$ and $\frac{q}{r} = \frac{673}{23}$. Thus let it be assumed $r = 23$ [and] it will be $p = 53$ and $q = 673 = s$ from which again it follows $\frac{a}{b} = \frac{23.673}{242.620}$ and $\frac{x}{y} = \frac{3.23}{2.53}$. Thus the four sought numbers will be

$$a = 23.673;\ b = 242.620;\ x = 3.23;\ y = 2.53.$$

Case III,
where $z = 0$ and $v = 1$.

§. 21. From the formulas given above for this case the following values are deduced:

$$z = 2;\ v = \frac{3n}{n+4};\ z = \frac{4(n-2)}{n+4};\ v = \frac{5nn - 24n + 16}{nn + 12n - 16},$$

where the first value of v itself is of no benefit; indeed the second $v = \frac{3n}{n+4}$ to which $z = 2$ corresponds, so that it will be $v - z = \frac{n-8}{n+4}$, [and it] gives

$$\frac{p}{r} = \frac{3tt}{tt + 4uu} \text{ and } \frac{q}{r} = \frac{tt - 8uu}{tt + 4uu}.$$

Therefore having assumed $r = tt + 4uu$ we will have $p = 3tt$ and $q = s = tt - 8uu$ from which we again deduce[n]:

$$\frac{a}{b} = \frac{t(tt - 8uu)}{4u(tt - 2uu)} \text{ and } \frac{x}{y} = \frac{(tt + 4uu)}{3tu}.$$

These formulas will be rendered more neatly, if in place of u we were to write $\frac{1}{2}u$, for then it will be:

$$\frac{a}{b} = \frac{t(tt - 2uu)}{u(2tt - uu)} \text{ and } \frac{x}{y} = \frac{2(tt + uu)}{3tu},$$

consequently our four sought numbers will be:

$$a = t(tt - 2uu);\ b = u(2tt - uu);\ x = 2(tt + uu);\ y = 3tu,$$

which case again agrees with the second particular solution.

Case IV,
where $z = 0$ and $v = -1$.

[m]The original mistakenly has 53 in place of 52 for the remainder of the section, which Euler apparently used, as the resulting values are consistent with this error (and do not have the desired property). The corrected values should be: $\frac{p}{r} = \frac{52}{23}, p = 52, a = 673, b = 6525, y = 2.52$.

[n]The original mistakenly has $t + 4uu$ in the numerator of $\frac{x}{y}$.

§. 22. From here therefore the derived values will proceed thus:

$$z = 0;\ v = -1;\ z = -2;\ v = \frac{-3n}{n-4};\ z = \frac{-4(n+2)}{n-4};$$

$$v = \frac{-(5nn + 24n + 16)}{nn - 12n - 16};$$

which values hardly differ from this preceding case, because having assumed n negative also v and z become negative. Hence if the letters p, q, r, s, are derived from the value $v = \frac{-3n}{n-4}$, by putting $n = \frac{tt}{uu}$, it will be

$$p = 3tt;\ q = s = tt + 8uu;\ r = tt - 4uu;$$

and if here as earlier we were to write $\frac{1}{2}u$ in place of u, the sought values of the letters a, b, x, y will be:

$$a = t(tt + 2uu);\ b = u(2tt + uu);\ x = 2(tt - uu);\ y = 3tu,$$

which formulas agree with the first particular case. But indeed the following values of v itself clearly will supply new solutions in all these cases, while not limited to particular cases, from which the generality of the new solution most clearly shines forth.

Case V,
which begins from $v = \infty$.

§. 23. Therefore from the general formulas the values for v and z will hold themselves successively thus:

$$v = \infty;\ z = \frac{-n}{2};\ v = \frac{4 - nn}{3n};\ z = \frac{16 - nn}{6n};\ v = \frac{n(64 - nn)}{8(nn + 8)}.$$

Here now the second value of v itself which is $\frac{4-nn}{3n}$, to which $z = \frac{-n}{2}$ corresponds, so that it is $v - z = \frac{8+nn}{6n}$, having put $n = \frac{tt}{uu}$, presents these values: $\frac{p}{r} = \frac{4u^4 - t^4}{3ttuu}$ and $\frac{q}{r} = \frac{8u^4 + t^4}{6ttuu}$, from which we deduce

$$p = 8u^4 - 2t^4;\ r = 6ttuu;\ q = s = 8u^4 + t^4,$$

from which we gather it to be

$$\frac{a}{b} = \frac{4u(8u^4 + t^4)}{t(16u^4 - t^4)} \text{ and } \frac{x}{y} = \frac{6t^3u}{8u^4 - 2t^4},$$

and thus it can be assumed

$$a = 4u(8u^4 + t^4);\ b = t(16u^4 - t^4);\ x = 6t^3u;\ y = 8u^4 - 2t^4.$$

Therefore let us adapt these formulas to certain special solutions, by assigning simpler values to the letters t and u, which solutions we describe in the following table:

t	1	2	3	3	1	1	2
u	1	1	1	2	3	2	3
a	12	1	4 . 89	8 . 11 . 19	11 . 12 . 59	8 . 43	3 . 83
b	5	0	3 . 5 . 13	3 . 7 . 25	5 . 7 . 37	5 . 17	2 . 5 . 8
x	1	2	81	2 . 81	9	2	3 . 6
y	1	1	7 . 11	17	17 . 19	21	7 . 11

§. 24. For these cases it was permitted to assign the letters z and v no more than a few terms, if indeed we are left an unrestricted value of the letter n; but if in its place we were to assume fixed squares, it is generally permitted to extend these series to several terms, which it will be helpful to show with some examples.

Explication of solutions
originating from the case $n = 4$.

§. 25. Thus in this case it will be $v' = \frac{z(z+8)}{2z+4} - v$ with keeping $z' = 2v - z$. In the explication of this, let us begin immediately at the fifth case where $v = \infty$, because we will soon see, that in it all the other four cases are included. Since thus for this case we have seen that it is $z = \frac{-n}{2}$ and following [it] $v = \frac{4-nn}{3n}$, the series of these letters will hold itself in the following way:

$v = \infty;$	$z = -2$	$v = +\frac{17}{3};$	$z = -\frac{14}{3}$
$v = -1;$	$z = 0$	$v = -\frac{11}{4};$	$z = -\frac{5}{6}$
$v = +1;$	$z = +2$	$v = +\frac{4}{21};$	$z = +\frac{17}{14}$
$v = \frac{3}{2};$	$z = +1$	$v = +\frac{31}{20};$	$z = +\frac{66}{35}$
$v = 0;$	$z = -1$	$v = +\frac{101}{119};$	$z = -\frac{16}{85}$
$v = -\frac{7}{2};$	$z = -6$	$v = -\frac{69}{55};$	$z = -\frac{434}{187}$
$v = +5;$	$z = +16$	$v = +\frac{741}{34};$	etc.

§. 26. Now here a solution to our question can be deduced from whatever value of v you wish when taken with the nearest z (for it is in the same way, whether it is joined with the preceding or with the following), when it is

$$\frac{p}{r} = v \text{ and } \frac{q}{r} = v - z,$$

and here again on account of $\sqrt{n} = 2$ it will be

$$\frac{a}{b} = \frac{4rs}{pp - qq} \text{ and } \frac{x}{y} = \frac{2rs}{pq}.$$

Thus having assumed $v = \frac{4}{21}$, to which $z = \frac{-5}{6}$ corresponds it will become[o] $v - z = \frac{43}{42}$. From here therefore it will be $\frac{p}{r} = \frac{4}{21}$ and $\frac{q}{r} = \frac{43}{42}$. Thus having taken $r = 42$ it will

[o]The original has $z - v$ here. Euler frequently changes signs and permutes letters when allowed.

be $p = 8$ and $q = s = 43$. From these values it is finally gathered $\frac{a}{b} = \frac{8.43}{5.17}$ and $\frac{x}{y} = \frac{21}{2}$, on account of which it will be

$$a = 8.43;\ b = 5.17;\ x = 21;\ y = 2.$$

From here it will be $ax = 3.7.8.43$ and $by = 2.5.17$, which have a common factor of 2. Therefore having put

$$2.3.7.43 = AB \text{ and } 5.17 = AA - BB,$$

and having assumed $A = 43$ and $B = 42$ it will become $AA - BB = 5.17$ as required, from which it will be $\sqrt{aaxx + bbyy} = 2(43^2 + 42^2) = 7226$ which number we have called z above. In a similar way it will be $ay = 16.43$ and $bx = 3.5.7.17$, and here we will have

$$AB = 8.43 \text{ and } A^2 - B^2 = 3.5.7.17.$$

Therefore having assumed $A = 43$ and $B = 8$ it will be $A^2 - B^2 = 35.51$ on account of which it will be $\sqrt{aayy + bbxx} = 1913$, which number we have indicated above by the letter v so that v and z become known. Moreover because the values $v = 0$ and $z = 0$ occur in the discovered series of letters z and v it is apparent that, all four prior cases are covered in this one.

Explication of solutions,
originating[p] from the case $n = \frac{1}{4}$.

§. 27. Thus in this case it will be $v' = \frac{2z(2z+1)}{8z+1} - v$ with keeping $z' = 2v - z$. For this [case] let us now run through all five cases established above[q]:

I.	II.	III.	IV.	V.
$v = 0$	$v = 0$	$v = \infty$	$z = 0$	$z = 0$
$z = 1$	$z = -1$	$z = -\frac{1}{8}$	$v = 1$	$v = -1$
$v = \frac{2}{3}$	$v = -\frac{2}{7}$	$v = \frac{21}{4}$	$z = 2$	$z = -2$
$z = \frac{1}{3}$	$z = +\frac{3}{7}$	$z = \frac{85}{8}$	$v = \frac{3}{17}$	$v = +\frac{1}{5}$
$v = -\frac{4}{11}$	$v = \frac{20}{31}$	$v = \frac{341}{32.43}$	$z = -\frac{28}{17}$	$z = \frac{12}{5}$
$z = \frac{35}{33}$	$z = \frac{187}{7.31}$	$z = \frac{6969}{16.43}$	$v = -\frac{55}{69}$	$v = \frac{119}{101}$

but it is apparent that, inverse values for v must be contained in the preceding case, since with the letters p and r permuted [then] in place of n ought to be written $\frac{1}{n}$.

§. 28. Therefore we do not mention here the special case where $n = 1$, since in the third particular case it has already been thoroughly exhausted. Moreover,

[p]The text has *oriundorum* here instead of *oriundarum* used earlier in the $n = 4$ case.

[q]The last entry of column II in the original lists the numerator incorrectly as 87, and the last entry in column III should have a numerator of -6969. Also, the order of the five cases has changed from when they were first introduced.

because in the case $n = 1$ the quadratic equation found between z and v comes out $(1+2z)vv - z(2+z)v + zz - 1 = 0$ or $(1-v)zz - 2v(1-v)z + vv - 1 = 0$, which equation, since it has the divisor $v - 1$, it is apparent that, having put $v = 1$ the corresponding value of z is left to our choice.

§. 29. At the place of the colophon[r] let us adjoin here a much more arduous problem, which I had scarcely dared to approach, if contrary to every expectation the third particular solution had not made its solution available.

Problem

To find four square numbers, aa, bb, cc, dd, of the innate character, that the product of whichever two at a time you wish, together with the product of the remaining two is a square, or so that these three formulas turn out to be squares:

$$\begin{aligned} aabb + ccdd &= \square \\ aacc + bbdd &= \square \\ aadd + bbcc &= \square. \end{aligned}$$

Solution.

§. 30. The third particular solution for the letters a, b, x, y, made these values available to us:

$$a = (t-u)(t+3u); \qquad b = 2(tt+3uu);$$
$$x = (t+u)(t-3u); \qquad y = 4tu;$$

where the formula found for x is thus similar to that found for a, so that with u having been accepted negative one is turned into the other. Wherefore if it is

$$bbxx + aayy = \square,$$

by permuting a and x even this formula $aabb + xxyy$ will be a square when from the condition of the problem already these two formulas:

$$aaxx + bbyy \text{ and } aayy + bbxx$$

are squares, and thus it is permitted to permute all these four letters a, b, x, y among themselves. Therefore there is no other need unless so that in place of x and y we write c and d, and a solution of this most general problem is contained in the following simple enough formulas:

$$a = 4tu; \qquad b = 2(tt+3uu); \;\; c = (t-u)(t+3u);$$
$$d = (t+u)(t-3u),$$

[r] the symbol marking the end of a book

where for the letters t and u it is permitted to take whatever numbers we desire[s].

§. 31. From here the simplest case arises by taking $t = 2$ and $u = 1$; for then it will be

$$a = 8;\ b = 14;\ c = 5;\ d = 3.$$

Moreover all the solutions conveyed in the particular solution equally satisfy those which we may place in the following table again[t] in full view:

t	2	1	4	1	4	5	2	5	4
u	1	2	1	4	3	2	5	4	5
a	8	8	16	16	48	40	40	80	80
b	14	26	38	98	86	74	158	146	182
c	5	7	21	39	13	33	51	17	19
d	3	15	5	55	35	7	91	63	99

A more succinct solution.

§. 32. Let two numbers f and g be sought, so that $ff + 3gg = hh$, which becomes, as we have seen $f = tt - 3uu, g = 2tu$, for then it will be $h = tt + 3uu$, and from here the four sought numbers will be

$$a = 2g;\ b = 2h;\ c = f + g;\ d = f - g,$$

from these values is hereafter obtained:

$$\begin{aligned} \sqrt{aabb + ccdd} &= ff + 7gg, \\ \sqrt{aacc + bbdd} &= 2(ff - fg + 2gg), \\ \sqrt{aadd + bbcc} &= 2(ff + fg + 2gg). \end{aligned}$$

From here it is clear that, in this solution it will always be $c - d = a$. Since indeed this solution offered itself from the preceding [ones] as if of its own will contrary to all expectation, let us adjoin the direct solution.

Direct solution.

§. 33. When division by the first term is done these three formulas must be squares:

$$1^\circ)\ 1 + \frac{ccdd}{aabb} = \square;\quad 2^\circ)\ 1 + \frac{bbdd}{aacc} = \square;\quad 3^\circ)\ 1 + \frac{bbcc}{aadd} = \square,$$

let it be put

$$\frac{cd}{ab} = \frac{pp - qq}{2pq} = P;\quad \frac{bd}{ac} = \frac{rr - ss}{2rs} = R;\quad \frac{bc}{ad} = \frac{tt - uu}{2tu} = T.$$

[s] *pro lubitu*

[t] See §. 13.

From these positions is now gathered

$$\frac{d}{a} = \sqrt{PR}; \quad \frac{c}{a} = \sqrt{PT}; \quad \frac{b}{a} = \sqrt{RT};$$

whereby toward finding a solution three formulas must be sought of the type P, R, T, so that the products from any two at a time are squares. For then the sought numbers a, b, c, d, will be easily confined in the integers. But such numbers are obtained, by taking

$$\begin{aligned} p &= 4fg,\ q = ff + 3gg;\ r = ff + 2fg - 3gg, \\ s &= ff + 3gg;\ t = ff - 2fg - 3gg;\ u = ff + 3gg, \end{aligned}$$

for then the solution given above will arise, which we may illustrate by a few examples.

Example 1.

§. 34. Let it be taken $p = 6; r = 5; t = 8;$

$$q = 1; s = 2; u = 7,$$

from these therefore it will be $P = \frac{5.7}{4.3}$; $R = \frac{3.7}{4.5}$; $T = \frac{3.5}{16.7}$ from which hereafter is gathered $\sqrt{PR} = \frac{7}{4} = \frac{d}{a}$; then indeed

$$\sqrt{PT} = \frac{5}{8} = \frac{c}{a}, \text{ and finally } \sqrt{RT} = \frac{3}{8} = \frac{b}{a}.$$

On account of this if we put $a = 8$ it will be $b = 3,\ c = 5,\ d = 14$, which is the simplest solution already unearthed above.

Example 2.

§. 35. Let it be taken $p = 6;\ r = 8;\ t = 27;$

$$q = 5;\ s = 3;\ u = 22,$$

from which it becomes $P = \frac{11}{4.3.5}$; $R = \frac{5.11}{16.3}$; $T = \frac{5.7^2}{4.3^3.11}$ from here farther on it follows that it will be $\sqrt{PR} = \frac{11}{24} = \frac{d}{a}$, $\sqrt{PT} = \frac{7}{36} = \frac{c}{a}$ and $\sqrt{RT} = \frac{35}{72} = \frac{b}{a}$. Therefore having taken $a = 72$ it will be $b = 35; c = 14; d = 33$, which values differ entirely from those which the previous method supplied, from whence it is clear that the above solution is not general but countless other solutions can have a place in addition, towards the finding of which a method of investigating suitable values for the three formulas P, R, T, in general is required, which task of unfolding I leave to others.