On finding three or more numbers, the sum of which is a square, while the sum of the squares is a fourth power* †

Leonhard Euler

§. 1.

Famous is the problem once proposed by Fermat[a], and recently studied eagerly by the distinguished Lagrange[b], in which are sought two positive integers, the sum of which is a square, while the sum of the squares is a fourth power. From here, I took the opportunity to extend this question to three or more numbers, relying on certain faith that its solution could be found without too much wandering. But after I had tried it, I soon discovered the very same difficulties that are involved in Fermat's problem. At last, I happily overcame all these obstacles, and indeed I obtained relatively small numbers satisfying the problem, whereas the smallest numbers satisfying Fermat's problem climb over a billion. Therefore the method which I used, I will propose here, after of course I convey in the interim the first examples, which generate the longest calculations.

§. 2. Let x, y, and z be three positive numbers, the sum of which is A^2, and the sum of the squares $xx + yy + zz = B^4$: and because the numbers are positive, it is immediately clear that $A^4 > B^4$, and therefore $A > B$, because A^4 includes the

*Originally published as *De tribus pluribusve numeris inveniendis, quorum summa sit quadratum, quadratorum vero summa biquadratum*, in *Mémoires de l'académie des sciences de St.-Petersbourg* 9, 1824, pp. 3-13. E763 in the Eneström index. Translated from the Latin by Christopher Goff, Department of Mathematics, University of the Pacific, Stockton CA 95204.

†Many thanks to Stacy Langton for pointing out errors in an earlier draft, and for shedding more light on Euler's vague references in sections 2 and 3, and to the referee for many valuable suggestions.

[a]The "famous" problem is to find integer sides of a right triangle in which the hypotenuse is a square, and the sum of the two legs is a square. Euler did not list the specific reference, but Lagrange did: Fermat's commentaries on Question 24, Book 6 of Diophantus. I found it in *Œuvres de Fermat*, Volume 1, XLIV, 336–339. Gauthier-Villars: Paris, 1891. Accessed on the Internet Archive, http://archive.org/details/oeuvresdefermat901ferm, 21 March 2012.

[b]*Sur quelques problèmes de l'analyse de Diophante*, 20 March 1777. In *Œuvres de Lagrange*, Volume 4, 377–398. Gauthier-Villars: Paris, 1869. Accessed on the Internet Archive, http://archive.org/details/uvresdelagrange04lagr, 21 March 2012.

squares themselves xx, yy, and zz, in addition to doubled products of the variables taken two at a time. Since, then, $x = A^2 - y - z$, I have put $y + z = p$ and $y - z = q$, from which it happens $yy + zz = \frac{pp+qq}{2}$. Therefore, because we have $x = A^2 - p$, the second equation will give

$$A^4 - 2A^2 p + pp + \frac{pp + qq}{2} = B^4,$$

from which we deduce $qq = 2(B^4 - A^4) + 4A^2 p - 3pp$, which formula can in no way be represented as a square, unless it consists of the one unique case, by which this may happen.[c]

§. 3. Because if the formula $2B^4 - 2A^4$ might turn out to be a square, which nevertheless is impossible[d], the thing might work with no difficulty. Therefore, the case remains[e], where $2B^4 - A^4$ becomes a square, for instance CC; for then will $qq = CC - A^4 + 4A^2 p - 3pp$, which, when reduced to $qq = CC - (AA - p)(AA - 3p)$, immediately gives: $q = C - v(AA - p)$, by which is found[f]

$$p = -\frac{2Cv + AA(1 + vv)}{3 + vv},$$

and having substituted this value, produces $q = \frac{3C - 2AAv - Cvv}{3+vv}$.

§. 4. But before all this, since certain values ought to be assigned to the two letters A and B so that $2B^4 - A^4 = CC$, we thus return to Fermat's problem.[g] So when such values are not able to be shown except among very large numbers, there is no hope of reaching solutions in small numbers by this method – Another way for us to understand these kinds of questions must be undertaken, one which is immune to such difficulties. But such an excellent way offered itself to me, optimal in its success, and in order to better show its power, I shall begin with Fermat's original problem.

Problem I.

To find two positive integers, x *and* y*, the sum of which is a square, and the sum of the squares of which is a fourth power.*

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[c]It seems likely that Euler is saying that the formula is not identically the square of a polynomial in A, B, and p, but that there may exist singular cases where the right hand side just so happens to equal a perfect square. Thanks to the referee for clarification.

[d]In Chapter 13 (paragraph 209) of the second part of his *Elements of Algebra*, Euler proves that the double of the difference of two fourth powers cannot be a square. This reference is mentioned by Lagrange (op. cit.).

[e]It's not clear why this is the remaining case. Perhaps it is just one remaining case, one which returns to Fermat's original problem.

[f]The corrected version of the following formula should be $\frac{-2Cv+AA(1+vv)}{3+vv}$.

[g]Lagrange (op. cit.) explicitly links this formulation to the original problem.

Solution:

§. 5. Let us begin with the latter condition. At first, indeed, the formula $xx + yy$ shall be rendered as a square, by placing $x = aa - bb$ and $y = 2ab$, for then $xx + yy = (aa + bb)^2$. In addition, then, this formula $aa + bb$ should be a square, which happens in the same way by setting $a = pp - qq$ and $b = 2pq$: from here, it follows that $xx + yy = (pp + qq)^4$, and thus the latter condition has now been fully satisfied. Then, it remains to satisfy the prior condition, namely that $x + y$ be a square.

§. 6. Thus, from these facts it is found that

$$x = aa - bb = p^4 - 6ppqq + q^4 \text{ and } y = 4p^3q - 4pq^3;$$

and so the following formula of fourth degree ought to be a square: $p^4 + 4p^3q - 6ppqq - 4pq^3 + q^4$, for which it must be noted both numbers p and q ought to be positive. It is also necessary that $p > q$, because otherwise the number y would become negative. It is also required that $a > b$, so that a positive number appears for x.

§. 7. The formula just found is resolved by putting the radical $\sqrt{x + y} = pp - 2pq + qq$, from which is computed $\frac{p}{q} = \frac{3}{2}$, or $p = 3$ and $q = 2$, which are positive numbers, and $p > q$. But because from here $a = 5$ and $b = 12$, a negative value results for x, and must be rejected. On account of this, a new method must be established, along known lines, and so in the end it remains that $q = 2$ but at the same time we set $p = 3 + v$, from which we deduce the following values:

$$\begin{aligned}
p^4 &= 81 + 108v + 54vv + 12v^3 + v^4, \\
4p^3q &= 216 + 216v + 72vv + 8v^3, \\
6p^2q^2 &= 216 + 144v + 24vv, \\
4pq^3 &= 96 + 32v, \\
q^4 &= 16.
\end{aligned}$$

and when the terms are collected, the formula given above takes on this form:

$$1 + 148v + 102vv + 20v^3 + v^4 = x + y,$$

the root of which, if it is stated $\sqrt{x + y} = 1 + 74v - vv$, leads to this equation: $1343 = 42v$, or $v = \frac{1343}{42}$; from which $p = 3 + v = \frac{1469}{42}$, with $q = 2$. Thus these letters, once they are made into whole numbers, will become $p = 1469$ and $q = 84$. From these, it now follows that $a = 1385.1553$ and $b = 168.1469$, or $a = 2150905$ and $b = 246792$. From this it is clear that because $a > b$, that both x and y are themselves positive, which, though they exceed a billion, are nevertheless the smallest satisfying the problem: These numbers are therefore

$$\begin{aligned} x &= 4{,}565{,}486{,}027{,}761 \\ y &= 1{,}061{,}652{,}293{,}520 \end{aligned}$$

which are the same that Fermat, and others after him, found.[h] The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017.

Problem II.

To find three numbers, whole, positive, x, y, z, *the sum of which is a square, while the sum of the squares is a fourth power.*

Solution:

§. 8. Let us begin again from the sum of squares, which is first rendered as a square, by placing $x = aa + bb - cc; y = 2ac; z = 2bc$; which thus will become $xx + yy + zz = (aa + bb + cc)^2$; whereby thus $aa + bb + cc$ ought to be made a square again, which will be done in a similar way by putting $a = pp + qq - rr; b = 2pr; c = 2qr$; for thus is obtained $xx + yy + zz = (pp + qq + rr)^4$; so that the latter condition is now fulfilled.

§. 9. Now let us express the letters x, y, z themselves through p, q, r:

$$\begin{aligned} x &= p^4 + q^4 + r^4 + 2ppqq + 2pprr - 6qqrr, \\ y &= 4qr(pp + qq - rr), \\ z &= 8pqrr. \end{aligned}$$

From here:

$$\begin{aligned} x + y + z &= p^4 + q^4 + r^4 + 2ppqq + 2pprr - 6qqrr \\ &+ 4ppqr + 4q^3r - 4qr^3 + 8pqrr, \end{aligned}$$

which form, arranged according to the powers of p, holds itself thus:

$$\begin{aligned} x + y + z &= p^4 + 2(q + r)^2pp + 8pqrr + q^4 + 4q^3r \\ &- 6qqrr - 4qr^3 + r^4, \end{aligned}$$

which must be represented as a square, while at the same time the single letters p, q, r, are positive, and $pp + qq > rr$. In addition, it is also necessary that when the values of the letters a, b, c, are compared, that $aa + bb > cc$.

[h] Incidentally, much of this story can be found in *Diophantus of Alexandria; a study in the history of Greek algebra*, by Sir Thomas L. Heath. Cambridge University Press: Cambridge, 1910, 293–300. Accessed on the Internet Archive, http://archive.org/details/diophantusofalex00heatiala, 21 March 2012. In particular, Heath explains Lagrange's proof of the minimality of Fermat's answers, and even provides (in a footnote on p. 299) a brief translation of a large portion of Euler's solution.

§. 10. Since in this formula, the third power of p is missing, the root in this situation could be $pp+(q+r)^2$. So now the fourth power as well as the second will be removed, and from the remaining terms it will be able to be defined $p = \frac{3}{2}q+r$, which value, on account of its simplicity, promises much more elegant solutions than we obtained in the previous problem. Now, on choosing $p = \frac{3}{2}q + r$, $a = \frac{13}{4}qq + 3qr; b = 3qr + 2rr; c = 2qr$; where now it is allowed that both letters q and r are chosen as you wish.

Example 1.

§. 11. Let us take $q = 2$ and $r = 1$, so that $p = 4$, then producing $a = 19; b = 8; c = 4$; from which the sought numbers themselves are deduced, which are $x = 409; y = 152; z = 64$. Of these numbers, the sum is $x+y+z = 625 = 25^2$; while the sum of the squares $x^2 + y^2 + z^2 = 194481 = 441^2 = 21^4$.

Example 2.

§. 12. Let us keep $q = 2$ and now also $r = 2$ is assumed, and [thus] $p = 5$; then at the same time $a = 25; b = 20; c = 8$. From here, the numbers themselves satisfying the question will be

$$x = 961;\ \ y = 400;\ \ z = 320,$$

the sum of which is $x+y+z = 1681 = 41^2$ and the sum of the squares $x^2+y^2+z^2 = 1185921 = 33^4$.

Another solution to the problem.

§. 13. While earlier we ordered the fourth-power formula according to powers of p, we now list this order according to the powers of the letter q, which after being done[i] will be $x + y + z =$

$$p^4 + 4q^3r + 2(pp - 3rr)qq + 4r(pp + 2pr - rr)q + (pp + rr)^2$$

the root of which, so that both prior terms are removed with the last, ought to be set to $qq + 2qr - pp - rr$; from which, the unfolding having been completed, is concluded

$$q = \frac{2pr(p + r)}{2rr - pp}$$

Example.

§. 14. Let $p = 1$ and $r = 1$, so that $q = 4$, and from here is computed $a = 16; b = 2; c = 8$, or being depressed by 2 $a = 8; b = 1; c = 4$; whence will further be $x = 49; y = 64; z = 8$. From here it happens

$$x + y + z = 11^2 \text{ and } xx + yy + zz = 9^4.$$

Without a doubt these numbers are the simplest satisfying the problem.

[i]The corrected version of the following formula should begin with q^4 rather than p^4.

Problem III.

To find four numbers x, y, z, v, *the sum of which is a square, while the sum of the squares is a fourth power.*

Solution:

§. 15. So that at first the sum of the squares be rendered as a square, let $x = aa + bb + cc - dd; y = 2ad; z = 2bd; v = 2cd$. For thus the sum of the squares will become $(aa + bb + cc + dd)^2$, the root of which is again rendered as a square, by placing $a = pp + qq + rr - ss; b = 2ps; c = 2qs; d = 2rs$. Lest now a very lengthy calculation proceed, on account of the multitude of terms, let us put for the sake of brevity $qq + rr - ss = A$, so that we have $a = pp + A$. From here it now follows that $x = p^4 + 2App + A^2 + 4p^2s^2 + 4q^2s^2 - 4r^2s^2; y = 4rspp + 4Ars; z = 8prss$ and $v = 8qrss$.

§. 16. Now the sum of the sought numbers, arranged according to the powers of p, will be:

$$p^4 + 2(A + 2ss + 2rs)p^2 + 8prss + A^2 + 4qqss - 4rrss + 4Ars + 8qrss,$$

and since it ought to be a square, its root is set

$$pp + (A + 2ss + 2rs);$$

which, after the substitution is done, will produce this equation:

$$2pr + qq + 2qr - 2rr - ss - 2rs - A = 0.$$

Therefore, having restored the value assumed in place of A, we will have $p = s + \frac{3}{2}r - q$, where now the letters q, r, s, can be taken as you please. Let us unfold some cases, being so careful that positive values are taken for q, r, s, that the value of x not become negative, which will easily happen provided that q is not taken very large[j].

Example 1.

§. 17. Let $r = 2$, $q = 1$, and $s = 1$, and so $p = 3$; from which now $a = 13; b = 6; c = 2; d = 4$, and from this are determined the sought numbers themselves $x = 193; y = 104; z = 48; v = 16$, the sum of which is $x + y + z + v = 361 = 19^2$; while the sum of the squares will be $xx + yy + zz + vv = (pp + qq + rr + ss)^4 = 15^4$.

Example 2.

[j]There are a lot of negatives in these last two sentences. At first glance, it seems that he is saying that we should be fine as long as we don't take q very large. But if q is large, then a and c are large, and thus x would be positive and there is no problem. So perhaps he's saying that x will be negative if we don't pick q large enough.

§. 18. Let us keep $r = 2$, but let $s = 1$ and $q = 2$, and so $p = 2$, from which is gathered $a = 11; b = 4; c = 4; d = 4$; and from here $x = 137; y = 88; z = 32; v = 32$, the sum of which $x + y + z + v = 289 = 17^2$, while the sum of the squares

$$x^2 + y^2 + z^2 + v^2 = 13^4.$$

Example 3.

§. 19. Keeping $r = 2$, let $s = 1$ and $q = 3$, and so $p = 1$. From here, the values of the letters a, b, c, d will be $a = 13; b = 2; c = 6; d = 4$; from which now $x = 193; y = 104; z = 16; v = 48$, and thus example 1 happens again.

In this way many such examples can be obtained with light work.

Problem IV.

To find five positive whole numbers x, y, z, v, w, *the sum of which is a square, while the sum of the squares is a fourth power.*

Solution:

§. 20. So that the sum of the squares is a square, let $x = aa+bb+cc+dd-ee; y = 2ae; z = 2be; v = 2ce; w = 2de$. So that it also produces a fourth power, now set $a = pp + qq + rr + ss - tt; b = 2pt; c = 2qt; d = 2rt; e = 2st$; but for the sake of brevity put $qq + rr + ss - tt = A$, so that $a = pp + A$, and from here it follows that $x = p^4 + 2App + A^2 + 4pptt + 4qqtt + 4rrtt - 4sstt; y = 4stpp + 4Ast; z = 8pstt; v = 8qstt; w = 8rstt$.

§. 21. Now the sum of the sought numbers, ordered according to the powers of p, is:

$$\begin{aligned} p^4 \quad &+ \quad 2pp(A + 2tt + 2st) + 8pstt + AA + 4qqtt \\ &+ \quad 4rrtt - 4sstt + 4Ast + 8qstt + 8rstt, \end{aligned}$$

the root of which is set to $pp + A + 2st + 2tt$; from which, after the square has been taken, the following equation results:

$$2ps + qq + 2qs + rr + 2rs - 2ss - tt - 2st - A = 0,$$

whence, once its value has been restored in place of A, produces $p = t + \frac{3}{2}s - r - q$; where now four numbers are regarded as needing to be assumed arbitrarily.

Example.

§. 22. Let $s = 2; t = 1; r = 1; q = 1$; and so $p = 2$. From here thus $a = 9; b = 4; c = 2; d = 2; e = 4$; and therefore the sought numbers will be $x = 89; y = 72; z = 32; v = 16; w = 16$, the sum of which $x + y + z + v + w = 225 = 15^2$, while the sum of the squares $x^2 + y^2 + z^2 + v^2 + w^2 = 11^4$.

And in the same way many sufficiently simple examples can be derived from our formulas.

Corollary.

§. 23. But if we consider the values found for the letter p, and compare them among themselves, from this a law opens out easily, by the power of which one will be allowed to progress to [the case of] many numbers, for instance:

For case 3 we found $p = r + \frac{3}{2}q$,

- - 4 - - $p = s + \frac{3}{2}r - q$,

- - 5 - - $p = t + \frac{3}{2}s - r - q$,

and thus for the case of six numbers it will be found that $p = u + \frac{3}{2}t - s - r - q$, and so forth, from which the general question, proposed for any number of numbers, must now be considered completely solved.

Comment.

§. 24. When in the first example of problem 2. the sum of the numbers themselves was found to be 25^2, which is also a fourth power, from here a new question can be formulated, concerning any number of numbers that must be found, the sum of which as well as the sum of the squares of which are fourth powers; certainly one will open this question soon to more careful consideration, that any solution found earlier can be adapted to this condition. Because if the sum of any number of numbers $x + y + z + \text{etc.} = A^2$ and the sum of the squares $x^2 + y^2 + z^2 + \text{etc.} = B^4$, then let the sought numbers themselves be set to $A^2x; A^2y; A^2z$; etc.; for then the sum of them will be $A^2.A^2 = A^4$, and thus a fourth power; the sum of the squares will certainly be $A^4.B^4$. But because in this way the sought numbers have a common factor among themselves, if this condition in addition be written, that the numbers to be found are to be prime among themselves [relatively prime] or have no common divisor; then the question must be considered not insufficiently difficult. Nevertheless, meanwhile in the following way even such questions can be resolved easily.

Problem V.

To find three positive numbers prime among themselves x, y, z, *the sum of which*[k] *as well as the sum of the squares of which are fourth powers.*

Solution:

§. 25. I posit, as in the second problem, $x = aa + bb - cc; y = 2ac; z = 2bc$, again let $a = pp + qq - rr; b = 2pr; c = 2qr$, and after the substitution has been done, the square root of the sum of those numbers themselves is set to $= pp + (q + r)^2$, and when we found above $p = r + \frac{3}{2}q$, then it is necessary that this expression $pp + (q + r)^2$ once more be represented as a square. Therefore let its root be established $p + \frac{f(q+r)}{g}$, and from here this equation will arise: $gg(q + r) = 2fgp + ff(q + r)$.

[k]The original "quoram" should probably be "quorum".

§. 26. Now let the found value $r + \frac{3}{2}q$ be written in place of p and the equation will take on this form: $(ff - gg)(q + r) + fg(2r + 3q) = 0$, from which is deduced[l] $\frac{q}{r} = \frac{ff+2fg-gg}{gg+3fg-ff}$. Therefore the solution of this problem is contained in this: Let $q = ff + 2fg - gg$ and $r = gg - 3fg - ff$, and so $p = \frac{1}{2}ff - \frac{1}{2}gg$, from which values first the letters a, b, c, and now from here the sought numbers x, y, z, can be formed in infinitely many ways.

§. 27. For example let $f = 1$ and $g = 3$, and so $q = 2$; $r = 1$ and $p = 4$. From here therefore we conclude $a = 19; b = 8; c = 4$, from which the sought numbers will be, $x = 409; y = 152; z = 63$, the sum of which is $x + y + z = 625 = 5^4$ and $x^2 + y^2 + z^2 = 21^4$.

§. 28. But first, the limits must be investigated, between which it is permitted to accept the letters f and g. Let the signs be changed in this limit and we will have $q = gg - 2fg - ff$ and $r = ff + 3fg - gg$, so that the values of which, as before, become positive, it ought to be that $\frac{g}{f} > 1 + \sqrt{2} > 2{,}414$; but so that r becomes positive, it ought to be $\frac{g}{f} < \frac{3+\sqrt{13}}{2} < 3{,}303$.[m] Thus let $f = 2$ and $g = 5$, and so $q = 1; r = 9; p = \frac{21}{2}$, or in integers $p = 21; q = 2; r = 18$; from which $a = 121; b = 756; c = 72$, and now from here $x = 580993; y = 17424; z = 108864$, the sum of which $x + y + z = 29^4$, while the sum of the squares

$$x^2 + y^2 + z^2 = 769^4.$$

§. 29. Thus in the same way it will be permitted to resolve this question for several sought numbers without difficulty; so I dwell on this argument no longer; for it will suffice to have explained the method of resolving all problems of this type comfortably and expeditiously.

[l]There is a typo in the denominator of the following formula, which should read $gg - 3fg - ff$.

[m]Euler omits the possibility that f and g have different signs, and lead to q and r both being negative. But a little algebra shows that if $fg < 0$ and the corresponding q and r are both negative, then switching f and g and changing the sign of one of them (so that both are positive) gives values within Euler's range, and ultimately lead to the same values for x, y and z.