A more accurate research
About
Brachistochrones
By the author
L. Euler.

§. 1. A principle of this kind, that I taught about these curves in book II of my Mechanicae[1], rests upon, what can't be allowed in a resistant medium. Thereafter, I tried to obtain the same argument from the first principles of Maxima and Minima in my isoperimetric treatment; to such a great degree are they, which I conducted there on the brachistochrone in a resistant medium, truly involved in the excessively generalised analytic formulae, such that thence barely anyone is able to pick out a true nature of those curves. On that account I decided to expand this same argument in a bigger study here and to derive it clearly and perspicuously from the first principles.

§. 2. From this principle I have consequently thereafter derived all brachistochrones in resistant media too. However, after a more fruitful isoperimetric theory was researched, I soon discovered that, which that principle in a resistant medium could not allow, nor did any of those things, which I investigated, studying all my mechanics work, yet concentrate on this defect, that I myself however happily corrected in my treatment of the isoperimetric problems, and I demonstrated determining the true brachistochrones for whichever resistant medium so much.

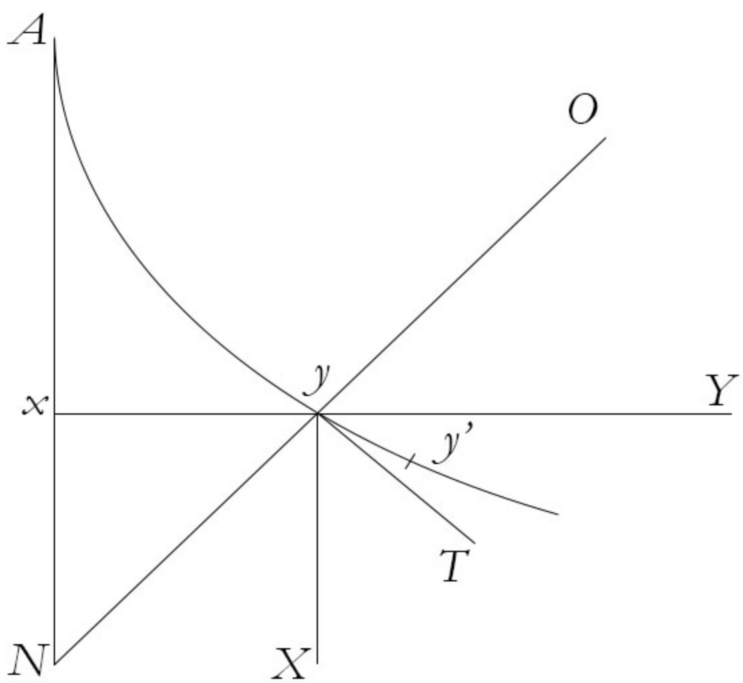
§. 3. That error, which I frankly admit, is meanwhile still not so enormous, that it can not only in a certain manner not be excused, but also united with truth, if only the state of the question is only altered briefly. Because if among all curves, over which a descending body acquires the same speed (of which the amount in any case still is infinite), and that the body is allowed to be led over from a higher end to a lower one, but only between them, intelligibly the one is sought, over which the body arrives from the highest point all the way to the lowest in the shortest time, then all brachistochrones, assigned by me and derived from the told principle, will be agreeing with the truth.

[1]Opera Omnia, II vol 2 p.170 §376, p.332 §673. Consult in addition the dissertation E042 of this volume, p.41

§. 4. Where it becomes however clearer, under which condition this principle has its place, and when it fails, I decide to develop a more accurate complete theory of brachistochrones. Since I observed, that the forces of this kind, to which this principle by no means can be adapted, can yet be presented, if also only the motion in vacuum is considered; on this account I will draw your attention away from all resistance here, since this argument is already sufficiently and courteously studied in my little isoperimetric work. On this account, I will not observe other forces, beyond such, that I called absolute, of which the action depends on a single place, on which the body moves, nor does its speed bring anything to the disturbing forces.

§. 5. This treatment is besides however divided in two parts, so that every motion of the body is either solved in the same plane, or it moves out of the same plane. Because for this distinction the method of finding brachistochrones should certainly be extended to different situations, although in the former case two coordinates, to introduce the calculus, suffice, in the latter inevitable case three coordinates are required, which new case is so much straight forward; neither do the brachistochrones, which don't continue in the same plane, come anyone to mind to investigate, as much as I indeed remember; on this account I will propose the treatment following this differential sequence bipartite.

I. On the Brachistochrones
Existing in the same plane.



§. **6.** Therefore, it is also necessary that all these disturbing forces, existing in the same plane, are, what I will however most generally consider. Let us therefore pose the motion of a body in the same plane, enclosed in the figure, and let Ay be the curve, over which the body moves, after it departed from the point A, which curve we refer to the axis Ax and we call the both coordinates $Ax = x$ and $xy = y$, let us call yy' in particular ds, such that thus, having posed $dy = pdx$, $ds = dx\sqrt{1+pp}$ holds; whence if yO were the radius of curvature of the curve, it will consistently hold that $yO = \frac{dx(1+pp)^{\frac{3}{2}}}{dp}$. The body is already disturbed by whichever forces in y, and it's always allowed to decompose them into both yX and yY, although they have the same directions as the coordinates. Let us therefore call these forces $yX = X$ and $yY = Y$, and because the action of those forces is assumed to depend on the unique locus of the body, y, it's as much allowed to consider that these letters X and Y are whichever functions of both coordinates x and y. I then consider those forces, which appear, if the true motoric forces are divided by the mass of the body and therefore are expressed by absolute numbers, already as much to be accelerative, having denoted the accelerative force of natural gravity, of which it's possible to compare all other forces, unit.

§. 7. When, while the body descends over the curve Ay, it then consequently sustains the action of the two forces $yX = X$ and $yY = Y$ in that place y, these forces unbind according to the direction of motion, or into the tangent yT and the direction normal to it yN, and the tangential force is found to be $yT = \frac{Xdx+Ydy}{ds}$, while the other, the normal force is surely $yN = \frac{Xdy-Ydx}{ds}$, by the former of which that motion of the body, proceding through the segment yy', will be accelerated, but the other normal force gives rise to the pressure, that the body exerts on the curve, if it's applied to the mass of the body, which, if the mass of the body is denoted by M, will be $\frac{M(Xdy-Ydx)}{ds}$, to which thus, according to the principle, which I established above, the centrifugal force, born from the curvature, of the body should be equal for brachistochrones.

§. 8. Let us now denote the speed, with which the body traverses the segment yy', with the letter v, which expresses the space, which will be traversed by this speed in a common second; and where we refer everything to the to be measured determinates, let g denote the altitude through which the mass firstly falls for a common second, and from the principles of motion it holds that $vdv = 2gTds$, if accordingly T denotes the tangential force, which was $\frac{Xdx+Ydy}{ds}$, from which this equation follows: $vdv = 2g\,(Xdx + Ydy)$; whence the determination of the speed depends on the integration of this formula, because it holds that $vv = 4g \int (Xdx + Ydy)$.

§. 9. Because if the letters X and Y will already be such functions of x, y, that this formula permits integration, which happens, as is the case, if $\frac{dX}{dy} = \frac{dY}{dx}$ will hold; then the speed of the body, v, will be a function directly determined by both variables x and y, and therefore it will depend on the sole location of the body, y. But if however this condition doesn't take place, then the speed will furthermore not depend on the sole location y, but besides involve the whole track, of the already traversed curve Ay, according to the values, which the formula receives through the whole traversed curve Ay by Xdx and Ydy; hence these two cases, to very carefully be mutually distinguished by each other, occur, such that naturally the integration formula $Xdx + Ydy$ is wide ranging. Soon the principle, told above, will however be accessible to take place in the sole former case, but it can surely by no means be evoked to be used in the other case.

§. 10. Because the tiny amount of time, in which the segment of the curve $yy' = ds = dx\sqrt{1+pp}$ is traversed, truly is so, such that the time through the curve Ay turns out to be minimal, or such that that curve is a true brachistochrone, it is necessary that the integral formula $\int \frac{ds}{v} = \int \frac{dx\sqrt{1+pp}}{v}$ obtains its minimal value between all curves, that are able to lead from the point A to the point y. In my isoperimetric treatment I however showed, if whichever integral formula $\int V dx$ should be either the maximum or the minimum, where V does not only depend on both those coordinates x and y in whichever way, but also on the relation between the differentials of those, and of which ordinate $dy = pdx$ holds, like already posed , as we already did, further $dp = qdx, dq = rdx, dr = sdx, etc.$ and it will hold that

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + etc.$$

when for the case of the maximum or minimum this equation always takes place:

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + etc. = 0$$

which equation then only thus takes place, when V will be a function of the quantities $x, y, p, q, r, etc.$, that is, when its value only depends on the sole point y and the segment of the curve in that locus. When however the function V furthermore involves whichever integral formulae, then, too, the ends, hence depending on that equation, should be added, in which case all the calculus demands most large digressions, which I will however not undertake here, but I will only stick to the equation, provided here.

§. 11. Hence it's then consequently apparent, that that equation of maximum or minimum can not take place, if the speed v isn't a function, determined by both x and y, or if the formula $\int (Xdx + Ydy)$ actually admits integration, which case I will therefore consider more accurately here. Because then consequently for our brachistochrones $\int V dx$ should become $\int \frac{dx\sqrt{1+pp}}{v}$, and thus $V = \frac{\sqrt{1+pp}}{v}$, dV will be $-\frac{dv}{vv}\sqrt{1+pp} + \frac{pdp}{v\sqrt{1+pp}}$, where instead of v it is thus necessary to substitute its value by x and y. Above, however, we had this equation: $vdv = 2g(Xdx + Ydy)$, whence dv is $\frac{2g}{v}(Xdx + Ydy)$, and just like that v is partly expressed by x and partly by y; on that account, if this value is substituted and a comparison is done with the general form, told above: $dV = Mdx + Ndy + Pdp + Qdq + etc.$ becomes

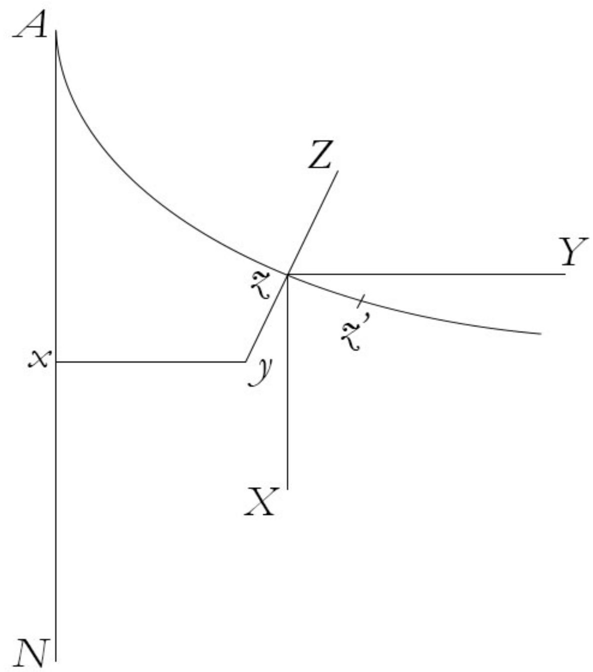
$$M = -\frac{2gX\sqrt{1+pp}}{v^3}; N = -\frac{2gY\sqrt{1+pp}}{v^3}; P = -\frac{p}{v\sqrt{1+pp}}; Q = 0; R = 0; etc.$$

and just like that we will now have this simple equation for the brachistochrone: $N - \frac{dP}{dx} = 0$, or $Ndx = dP$, such that the value of that P should therefore already again be differentiated. dP will moreover be $-\frac{dv}{vv} \cdot \frac{p}{\sqrt{1+pp}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}}$, and for that reason $dP = -\frac{2g(Xdx+Ydy)}{v^3} \cdot \frac{p}{\sqrt{1+pp}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}}$, to which expression the quantity $Ndx = -\frac{2gYdx\sqrt{1+pp}}{v^3}$ should be equal, from which equation further is acquired, that will hold: $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2gXdx}{v^3} \cdot \frac{p}{\sqrt{1+pp}} - \frac{2gYdx}{v^3\sqrt{1+pp}}$ or $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2g}{v^3\sqrt{1+pp}} (Xdy - Ydx)$.

§. 12. Moveover we invented above, that the normal force, to be born from the disturbing forces and pressing along yN, is $\frac{Xdy-Ydx}{ds}$, which equation of ours, if it's called Θ, such that $\Theta = \frac{Xdy-Ydx}{dx\sqrt{1+pp}}$, will be discovered $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2g\Theta dx}{v^3}$, and thus Θ will be $\frac{vv}{2gdx} d \cdot \frac{p}{\sqrt{1+pp}}$. Truly, it holds $d \cdot \frac{p}{\sqrt{1+pp}} = \frac{dp}{(1+pp)^{\frac{3}{2}}}$, and thus Θ will become $\frac{vv}{2gdx} \cdot \frac{dp}{(1+pp)^{\frac{3}{2}}}$. We saw however further that the radius of curvature in the point y is $\frac{dx(1+pp)^{\frac{3}{2}}}{dp}$, which, if it's called r, will make $\Theta = \frac{vv}{2gr}$. It is moreover further true that this formula $\frac{vv}{2gr}$ expresses the centrifugal force, with which the curve in the point y is pressed by a body, descending along that curvature, which force we thus now observe to be equal to the normal force Θ, whenever the formula $\int (Xdx + Ydy)$ permits integration, contrary to that the equation for a brachistochrone must otherwise surely very much have itself, and of which the determination requires most intricate calculi. Conveniently however, it comes with experience, whenever a body is disturbed by real forces, of which kind gravity is, and whichever centripetal forces and however many, disturbing according to whichever functions of distance, such that the formula $\int (Xdx + Ydy)$ permits integration and for that reason the principle established above actually takes place. Only imaginary forces are certainly excluded, which indeed can find whichever place in the nature of matters.

II. On the Brachistochrones

Not existing in the same plane.



§. **13.** This case occurs, when forces, by which a body is simultaneously disturbed, won't be situated in the same plane. Let henceforth the curve Az be the sought brachistochrone, over which a body will begin to be moved from the point A. Let us therefore determine whichever its point z by the three coordinates, which are $Ax = x; xy = y; yz = z$; let a segment of the curve verily be called $zz' = ds$; such that so ds^2 is $dx^2 + dy^2 + dz^2$. Moreover, the disturbing forces, whenever they will be compared, are decomposed in the same three fixed directions and are called $zX = X; zY = Y; zZ = Z$; which quantities thus can be whichever functions of the three variables x, y, z.

§. **14.** To already define the motion of the curve, let's define the whole matter from the first principles of motion, and, after posing a segment of time $= dt$, the determination of the motion of the body is contained in these three formulae:

$$1\,°)\ \frac{ddx}{dt^2} = 2gX;\ 2\,°)\ \frac{ddy}{dt^2} = 2gY;\ 3\,°)\ \frac{ddz}{dt^2} = 2gZ;$$

where g again describes the altitude of the fall of a mass in the first common second since we want to express the time t in common seconds. Now, the first of these equations multiplied by dx, the second by dy, the third by dz and integrated, they yield:

$$\frac{dx^2+dy^2+dz^2}{dt^2}=4g\int(Xdx+Ydy+Zdz)$$

which equation, on account of $dx^2+dy^2+dz^2=ds^2$, is reduced to this:

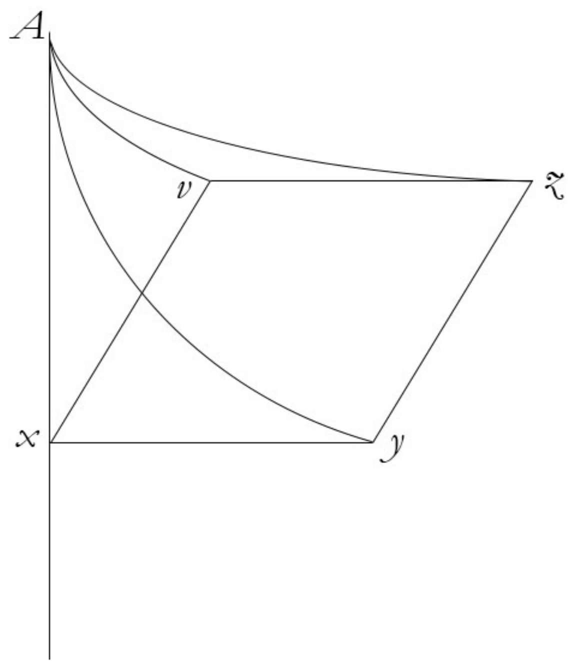
$$\frac{ds^2}{dt^2}=4g\int(Xdx+Ydy+Zdz)$$

Hence, because $\frac{ds}{dt}$ expresses the speed, with which the body traverses the segment zz', if it is posed $=v$, we will have this determination for it: $vv=4g\int(Xdx+Ydy+Zdz)$, whence it follows that this will hold:

$$vdv=2g\,(Xdx+Ydy+Zdz)$$

§. 15. Moreover, from these differentio-differential formulae it will too follow that this integrand derives: $\frac{yddx-xddy}{dt^2}=2g\,(yX-xY)$ of which the integral will be $\frac{ydx-xdy}{dt}=2g\int(yX-xY)\,dt$. Because we just discovered $\frac{ds^2}{dt^2}=vv$, we write $\frac{ds^2}{vv}$ instead of dt^2, and it holds that $\frac{ydx-xdy}{ds}=\frac{2g}{v}\int(Xy-Yx)\,\frac{ds}{v}$. In the same way we will learn that $\frac{zdx-xdz}{ds}=\frac{2g}{v}\int(Xz-Zx)\,\frac{ds}{v}$, and lastly $\frac{zdy-ydz}{ds}=\frac{2g}{v}\int(Yz-Zy)\,\frac{ds}{v}$. And it will help to have noted these formulae in the following one.

§. 16. Already having invented the speed of a body, such relation between the three coordinates x, y and z must be investigated, that the time, in which the arc of the curve Az is traversed, becomes minimal for all. In this matter thus we must return to the isoperimetric method. But this method is verily accommodated to only two variables, like how I truly managed it; meanwhile this question, too is yet able to be reduced to the case of two variables, since we call those to help, which are taught from the projections of the curves, that aren't situated in the same plane.



§. 17. Let us then consequently consider the projection of our curve Az, made in the plane of the table, which is Ay, of which thus the nature is expressed by an equation between both variables x and y, for which we state $dy = pdx$ and it will hold that an element of this projection $= dx\sqrt{1+pp}$. Let in a similar way Av be the projection of our curve, constructed in the plane, normal to the table, above the axis Ax, of which the nature is expressed by an equation between both variables $Ax = x$ and $xv = yz = z$, for which we pose $dz = qdx$, such that an element of this projection is $dx\sqrt{1+qq}$. Moreover, it is evident that an element of the true curve Az will be $= ds = dx\sqrt{1+pp+qq}$. Let us call the prior projection Ay 'lying', and the other Av 'upright'.

§. 18. It is however manifest that, if both these projections will be found, from joining them the same curve Az can most easily be determined. Because the abscissa $Ax = x$ is truly common to both projections, if we erect yz, itself being equal to xv, perpendicular from the point y, the point z will be in this same sought curve. Yet, one of those projections doesn't accomplish a matter by any means, because as the lying projection is able to meet together with infinitely many curves, so does the upright one.

§. 19. Having noted this well, the whole question of the minimum sought will be thus established bipartite. Let us first of course observe the upright projection as given, and between all curves, to which the same upright projection responds, we seek the one, in which the integral formula $\int \frac{ds}{v}$ obtains the minimum value, that which by only two coordinates will be possible to be provided. Because truly the upright projection Axv is observed as given, it will be possible to consider it, applied to its z, as a function of the abscissa x, and in the same way the quantity $q = \frac{dx}{dz}$ too will be a function of that x, and if we apply the isoperimetric precept to this case, we will discover that one, for which the formula $\int \frac{ds}{v}$ obtains a minimum value, between all curves, having the same upright projection.

§. 20. In the same way, the lying projection Axy will be considered as noted, and between all curves that have this projection in common, the one, for which the same formula $\int \frac{ds}{v}$ obtains a minimum value, is sought by the same method of maxima and minima, and now in this investigation, both y and $p = \frac{dy}{dx}$ can be had for functions of x, such as thus only both remaining x and z should already be counted as variables again, and the calculus by the same precept and before will be possible to be procured, if we just write z instead of y and q instead of p.

§. 21. But if in this way already we invented a curve of minimum both between all curves having the same upright projection, and between all curves having the same lying one, since for the former a certain equation came forth between x and y, for the other surely an equation between x and z, these two determinations, taken together, will provide a true brachistochrone, between all intelligibly possible curves.

§. 22. According to that precept it will already be easy to pick out brachistochrones, or those curves, in which the formula $\int \frac{dx\sqrt{1+pp+qq}}{v}$ assumes a minimum value. Moreover, like before, it is necessary, that v is a function determined by the variables x, y, z, that which is unable to occur, if the formula $\int (Xdx + Ydy + Zdz) = \frac{vv}{4g}$ does not allow integration; on that account we here treat only those cases. Then hence consequently vdv will be $2g\,(Xdx + Ydy + Zdz)$, and for that reason $dv = \frac{2g}{v}\,(Xdx + Ydy + Zdz)$. Let us thus first observe the upright projection as given, such that therefore both z and q are functions of only x; whence if we pose

$$d \cdot \frac{\sqrt{1+pp+qq}}{v} = Mdx + Ndy + Pdp$$

The equation for the sought curve will be $Ndx - dP = 0$, where it conveniently occurs, that the quantity M does not enter in that equation.

§. 23. Since we therefore do not engage in the quantity M, only two variables come in the computation in this differentiation, of course y and p, since z and q are had for functions of x, and the differentials of these are contained in the portion Mdx, which we're allowed to remove. By these means it is necessary that the values of the letters N and P are sought by differentiation, and since the quantity p does not enter in the speed v, for the portion Pdp, whence at once $P = \frac{p}{v\sqrt{1+pp+qq}}$ appears.

§. 24. Then consequently the variable v rests, which is possible to be considered as a function of only that y, and just like that for our present use dv will be $\frac{2gYdy}{v}$, and for that reason $d \cdot \frac{1}{v} = -\frac{2gYdy}{v^3}$, and thus N will be $-\frac{2gY}{v^3} \cdot \sqrt{1+pp+qq}$. Hence the sought equation is thus elicited:

$$+\frac{2gYdx}{v^3}\sqrt{1+pp+qq} + d \cdot \frac{p}{v\sqrt{1+pp+qq}} = 0$$

§. 25. In a similar way, if we assume the lying projection for known, such that y and p are already functions of only that x, the equation discovered before is transfered to this case, if only the letters y and z and likewise p and q are mutually permuted. In this way this equation comes forth:

$$\frac{2gZdx}{v^3}\sqrt{1+pp+qq} + d \cdot \frac{q}{v\sqrt{1+pp+qq}} = 0$$

which equation, connected with the previous, will determine the same sought brachistochrone, seeing that its determination requires two equations, for that reason, because both remaining y and z should be defined by the abscissa x anywhere.

§. 26. Behold, thus the resolution to our problem is contained in these two equations:

$$\frac{2gYdx}{v^3} \cdot \sqrt{1+pp+qq} + d \cdot \frac{p}{v\sqrt{1+pp+qq}} = 0$$

$$\frac{2gZdx}{v^3} \cdot \sqrt{1+pp+qq} + d \cdot \frac{q}{v\sqrt{1+pp+qq}} = 0$$

where all quantities for the variables must be already intelligibly had. Moreover, it fits that the posterior formulae here evolved somewhat more with help of this reduction:

$$d\cdot\frac{p}{v\sqrt{1+pp+qq}}=-\frac{dv}{vv}\cdot\frac{p}{\sqrt{1+pp+qq}}+\frac{1}{v}d\cdot\frac{p}{\sqrt{1+pp+qq}}$$

Now moreover on account of $dv=\frac{2g(Xdx+Ydy+Zdz)}{v}$ $\frac{dv}{vv}$ will be $+\frac{2g(Xdx+Ydy+Zdz)}{v^3}$, and hence our two equations assume the following forms.

$$\frac{2gYdx}{v^3}\sqrt{1+pp+qq}-\frac{2g\left(Xdx+Ydy+Zdz\right)}{v^3}\cdot\frac{p}{\sqrt{1+pp+qq}}+\frac{1}{v}d\cdot\frac{p}{\sqrt{1+pp+qq}}=0$$

$$\frac{2gZdx}{v^3}\sqrt{1+pp+qq}-\frac{2g\left(Xdx+Ydy+Zdz\right)}{v^3}\cdot\frac{q}{\sqrt{1+pp+qq}}+\frac{1}{v}d\cdot\frac{q}{\sqrt{1+pp+qq}}=0$$

These equations are multiplied by $\frac{v^3}{2g}$ and the parts, reduced prior to the denominator $\sqrt{1+pp+qq}$, are restored in the following way:

$$\frac{\left(Y\left(1+qq\right)-pX\right)dx-pZdz}{\sqrt{1+pp+qq}}+\frac{vv}{2g}d\cdot\frac{p}{\sqrt{1+pp+qq}}=0$$

$$\frac{\left(Z\left(1+pp\right)-qX\right)dx-qYdy}{\sqrt{1+pp+qq}}+\frac{vv}{2g}d\cdot\frac{q}{\sqrt{1+pp+qq}}=0$$

which equations onwards, on account of $dy=pdx$ and $dz=qdx$, will in this way be transformed:

$$\frac{Y\left(1+qq\right)-pX-pqZ}{\sqrt{1+pp+qq}}+\frac{vv}{2gdx}d\cdot\frac{p}{\sqrt{1+pp+qq}}=0$$

$$\frac{Z\left(1+pp\right)-qX-pqY}{\sqrt{1+pp+qq}}+\frac{vv}{2gdx}d\cdot\frac{q}{\sqrt{1+pp+qq}}=0$$

Because, if we delete the terms, containing z and q, here, the equation, invented for the preceding case, appears from the prior evident equation, from it surely produces:

$$\frac{Xp-Y}{\sqrt{1+pp}}=\frac{vv}{2g}d\cdot\frac{p}{\sqrt{1+pp}}$$

which equation excellently convenes with the above discovered; the posterior equation verily intelligibly disappears in this case.

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