Research into some Remarkable Integrations in the Analysis of Functions with Two Variables known as Partial Differences

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Let z be any function of two variables x and y, one knows that the first differentiation, depending on taking only either the variable x or the variable y, yields to the first degree differential formulas: $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. The second differential yields to these three second order differential formulas: $\frac{\partial\partial z}{\partial x^2}, \frac{\partial\partial z}{\partial x\partial y}, \frac{\partial\partial z}{\partial y^2}$. The third differential yields to these four third order differential formulas : $\frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial x^2\partial y}, \frac{\partial^3 z}{\partial x\partial y^2}, \frac{\partial^3 z}{\partial y^3}$. The fourth differential yields to these five fourth order differential formulas: $\frac{\partial^4 z}{\partial x^4}, \frac{\partial^4 z}{\partial x^3\partial y}, \frac{\partial^4 z}{\partial x^2\partial y^2}, \frac{\partial^4 z}{\partial x\partial y^3}, \frac{\partial^4 z}{\partial y^4}$; and so on. We are dropping here the brackets, within which we usually put those formulas, as there is no ambiguity to worry about in the researches that we are doing.

Having set that up, I will consider here the following expressions:

1. $P = x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial x}$
2. $Q = x^2.\frac{\partial\partial z}{\partial x^2} + xy.\frac{\partial\partial z}{\partial x\partial y} + y^2.\frac{\partial\partial z}{\partial y^2}$
3. $R = x^3.\frac{\partial^3 z}{\partial x^3} + 3xxy.\frac{\partial^3 z}{\partial x^2\partial y} + 3xyy.\frac{\partial^3 z}{\partial x\partial y^2} + y^3.\frac{\partial^3 z}{\partial y^3}$
4. $S = x^4.\frac{\partial^4 z}{\partial x^4} + 4x^3y.\frac{\partial^4 z}{\partial x^3\partial y} + 6xxyy.\frac{\partial^4 z}{\partial x^2\partial y^2} + 4xy^3.\frac{\partial^4 z}{\partial x\partial y^3} + y^4.\frac{\partial^4 z}{\partial y^4}$

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and so on. In general we will have the following:

$$Z = x^\lambda.\frac{\partial^\partial z}{\partial x^\lambda}+\frac{\lambda}{1}x^{\lambda-1}y.\frac{\partial^\lambda z}{\partial x^{\lambda-1}\partial y}+\frac{\lambda}{1}.\frac{\lambda-1}{2}.x^{\lambda-2}y^2.\frac{\partial^\lambda z}{\partial x^{\lambda-2}\partial y^2}+\frac{\lambda}{1}.\frac{\lambda-1}{2}.\frac{\lambda-2}{3}.\frac{\partial^\lambda z}{\partial x^{\lambda-3}\partial y^3}+\dots$$

Here I first observe, that each of these expressions can be immediately obtained from the preceding one, and we will see that we will always have:

1. $Q = x.\frac{\partial P}{\partial x} + y.\frac{\partial P}{\partial y} - 1.P$
2. $R = x.\frac{\partial Q}{\partial x} + y.\frac{\partial Q}{\partial y} - 2.Q$
3. $S = x.\frac{\partial R}{\partial x} + y.\frac{\partial R}{\partial y} - 3.R$
4. $T = x.\frac{\partial S}{\partial x} + y.\frac{\partial S}{\partial y} - 4.S$

and so on. Where we have to notice that if we put O for the formula preceding the first one P, we will have O=z; giving $P = x.\frac{\partial O}{\partial x} + y.\frac{\partial O}{\partial y} - 0.O$.

To prove the truth of all these equations, let's start by the first one, which expresses the value of Q, and since $P = x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y}$, the differentiation will give us

$$\frac{\partial P}{\partial x} = I.\frac{\partial z}{\partial x} + x.\frac{\partial\partial z}{\partial x^2} + y.\frac{\partial\partial z}{\partial x\partial y} \qquad and$$

$$\frac{\partial P}{\partial y} = I.\frac{\partial z}{\partial y} + x.\frac{\partial\partial z}{\partial x\partial y} + y.\frac{\partial\partial z}{\partial y^2}.$$

From there, we will get this equation:

$$x.\frac{\partial P}{\partial x} + y.\frac{\partial P}{\partial y} = x\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} + xx\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + yy\frac{\partial\partial z}{\partial y^2}$$

which easily reduces to this form: $x.\frac{\partial P}{\partial x} + y.\frac{\partial P}{\partial y} = P + Q$, and moving forward we get

$$Q = x.\frac{\partial P}{\partial x} + y.\frac{\partial P}{\partial y} - P.$$

For the second of our equations, since we have supposed

$$Q = x^2\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + y^2\frac{\partial\partial z}{\partial y^2},$$

we get

$$\frac{\partial Q}{\partial x} = 2x\frac{\partial\partial z}{\partial x^2} + 2y\frac{\partial\partial z}{\partial x\partial y} + x^2\frac{\partial^3 z}{\partial x^3} + 2xy\frac{\partial^3 z}{\partial x^2\partial y} + y^2\frac{\partial^3 z}{\partial x\partial y^2};$$

$$\frac{\partial Q}{\partial y} = 2x\frac{\partial\partial z}{\partial x\partial y} + 2y\frac{\partial\partial z}{\partial y^2} + x^2\frac{\partial^3 z}{\partial x^2\partial y} + 2xy\frac{\partial^3 z}{\partial x\partial y^2} + y^2\frac{\partial^3 z}{\partial y^3}.$$

Now the combination of these formulas will give:

$$x.\frac{\partial Q}{\partial x}+y.\frac{\partial Q}{\partial y} = 2xx.\frac{\partial\partial z}{\partial x^2}+4xy.\frac{\partial\partial z}{\partial x\partial y}+2yy.\frac{\partial\partial z}{\partial y^2}+x^3.\frac{\partial^3 z}{\partial x^3}+3xxy.\frac{\partial^3 z}{\partial x^2\partial y}+3xyy.\frac{\partial^3 z}{\partial x\partial y^2}+y^3\frac{\partial^3 z}{\partial y^3}.$$

This equation obviously reduces to the following one: $x.\frac{\partial Q}{\partial x} + y.\frac{\partial Q}{\partial y} = 2Q + R$, in such a way that $R = x.\frac{\partial Q}{\partial x} + y.\frac{\partial Q}{\partial y} - 2Q$.

To show the truth of our third equation, since we have

$$R = x^3.\frac{\partial^3 z}{\partial x^3} + 3xxy.\frac{\partial^3 z}{\partial x^2\partial y} + 3xyy.\frac{\partial^3 z}{\partial x\partial y^2} + y^3\frac{\partial^3 z}{\partial y^3},$$

we then get :

$$\frac{\partial R}{\partial x} = 3xx.\frac{\partial^3 z}{\partial x^3}+6xy.\frac{\partial^3 z}{\partial x^2\partial y}+3yy.\frac{\partial^3 z}{\partial x\partial y^2}+x^3\frac{\partial^4 z}{\partial x^4}+3xxy.\frac{\partial^4 z}{\partial x^3\partial y}+3xyy.\frac{\partial^4 z}{\partial x^2\partial y^2}+y^3\frac{\partial^4 z}{\partial x\partial y^3};$$

$$\frac{\partial R}{\partial y} = 3xx.\frac{\partial^3 z}{\partial x^2\partial y}+6xy.\frac{\partial^3 z}{\partial x\partial y^2}+3yy.\frac{\partial^3 z}{\partial y^3}+x^3\frac{\partial^4 z}{\partial x^3\partial y}+3xxy.\frac{\partial^4 z}{\partial x^2\partial y^2}+3xyy.\frac{\partial^4 z}{\partial x\partial y^3}+y^3\frac{\partial^4 z}{\partial y^4};$$

These two equations combined give:

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = 3x^3.\frac{\partial^3 z}{\partial x^3} + 9xxy.\frac{\partial^3 z}{\partial x^2\partial y} + 9xyy.\frac{\partial^3 z}{\partial x\partial y^2} + 3y^3\frac{\partial^3 z}{\partial y^3} + x^4.\frac{\partial^4 z}{\partial x^4}+$$

$$4x^3y.\frac{\partial^4 z}{\partial x^3\partial y} + 6xxyy.\frac{\partial^4 z}{\partial x^2\partial y^2} + 4xy^3\frac{\partial^4 z}{\partial x\partial y^3} + y^4\frac{\partial^4 z}{\partial y^4};$$

equation that obviously reduces again to this one:

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = 3R + S,$$

where one has therefore

$$S = x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} - 3R.$$

It will be superfluous to show by the same computations the truth of the following equations, since it is already clear that similar operations will always yield to equations as assigned above. But these beautiful relations between the quantities $P, Q, R, ...$ will take us to the advantage of finding the integrals, and even complete integrals of the following differential equations: 1) $P = 0$; 2) $Q = 0$; 3) $R = 0$; 4) $S = 0$; and so on. For this purpose, we just have to solve the three following preliminary problems.

PRELIMINARY PROBLEM I

Find a function of two variables x and y, which is v, such that it becomes

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0.$$

SOLUTION

Since v is function of x and y, assume that differentiating it, taking both x and y as variables, one finds $\partial v = p\partial x + q\partial y$, in such a way that $p=\frac{\partial v}{\partial x}$ and $q=\frac{\partial v}{\partial y}$, and subject to satisfy the equation: $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=xp+yq=0$, from which one has $q=-\frac{xp}{y.}$ This value being substituted, it will give $\partial v = p\partial x - \frac{px\partial y}{y} = p(\frac{y\partial x - x\partial y}{y})$. So this formula has to be integrable. Thus one reduces it to the form: $\partial v = p(\frac{y\partial x - x\partial z}{yy})$, where setting $\frac{x}{y}=p$, to have $py\partial t = \partial v$, it is clear that for this formula to be integrable, py has to be a function of the only variable t, and thus the integral will be a function of the same quantity t.

Let's use in the following for arbitrary functions , the symbols ***U,B,C,D*** and so on such that $\boldsymbol{U}$:t or $\boldsymbol{B}$:t, or $\boldsymbol{C}$:t represents an arbitrary function of t. In addition of that we will use as in general, for differential of any order : $\partial.\boldsymbol{U} : t = \partial t \boldsymbol{U'} : t$, $\partial.\boldsymbol{U'} : t = \partial t \boldsymbol{U''} : t$, $\partial.\boldsymbol{U''} : t = \partial t \boldsymbol{U'''} : t$, and so on. Having noticed that, our last equation integrated gives $v = \boldsymbol{U} : t$, or because of $t=\frac{x}{y}$, we will have $v = \boldsymbol{U} : \frac{x}{y}$; in such a way that one can take as v any function of $\frac{x}{y}$; where it is good to notice that all these functions are understood as homogeneous functions of zero dimension of x and y :

PRELIMINARY PROBLEM II

Find a function of two variables x and y, which is V, such that there is

$$nv = x\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}$$

SOLUTION

Let, as before, $\partial v = p\partial x + q\partial y$, and since $p=\frac{\partial v}{\partial x}$ et $q=\frac{\partial v}{\partial y}$, we will this condition to be satisfied $nv = px+qy$. Let's eliminate the letter q in these two equations, by multiplying the first by y and the other by ∂y, and by taking off the last from the first, we will have this one: $y\partial v - nv\partial y = p(y\partial x - x\partial y)$; where one must find the function p, for this equation to become integrable.

To turn the first part of this equation into an integrable one, It's enough to divide it by y^{n+1}, from which one gets

$$\frac{y\partial v - nv\partial y}{y^{n+1}} = \partial.\frac{v}{yn} = p(\frac{y\partial x - x\partial y}{y^{n+1}}).$$

But since the formula $\frac{y\partial x - x\partial y}{y^2}$ is the differential of $\frac{x}{y}$, Let's represent our equation under the form:

$$\partial.\frac{v}{y^n} = \frac{p}{y^{n-1}}.\frac{y\partial x - x\partial y}{yy} = \frac{p}{y^{n+1}}\partial.\frac{x}{y};$$

where it is obvious that $\frac{p}{y^{n-1}}$ must be function of $\frac{x}{y}$; and since the integral will be therefore such a function as well, we will have, integrating this equation, $\frac{v}{y^n} = \boldsymbol{U} : \frac{x}{y}$; from which we will get this value for the function needed: $v = y^n \boldsymbol{U} : \frac{x}{y}$.

Since a function of $\frac{x}{y}$, being multiplied by $\frac{x}{y}$, or in general by $\frac{x^n}{y^n}$, always remains a function of $\frac{x}{y}$, instead of $\boldsymbol{U} : \frac{x}{y}$ we will be able to write $\frac{x^n}{y^n}\boldsymbol{B} : \frac{x}{y}$, and thus the found value for v will be able to be expressed by $v = x^n\boldsymbol{B} : \frac{x}{y}$ or $v = x^{n-1}y\boldsymbol{B} : \frac{x}{y}$, or $v = x^{n-2}y^2\boldsymbol{B} : \frac{x}{y}$, and so on. But one knows that all these functions are called homogeneous, which the number of dimensions is everywhere equal to n.

PRELIMINARY PROBLEM III

Find a function of two variables x and y, which is V, such that there is

$$nv = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - y^\lambda \boldsymbol{U} : \frac{x}{y}.$$

SOLUTION

Let again $dv = p\partial x + q\partial y$, to have $p = \frac{\partial v}{\partial x}$ and $q = \frac{\partial v}{\partial y}$, and one will this condition to be satisfied: $nv = px + qy + y^\lambda \boldsymbol{U} : \frac{x}{y}$. Now from these two equations, one forms this one:

$$y\partial v - nv\partial y = p(y\partial x - x\partial y) - y^\lambda \partial y \boldsymbol{U} : \frac{x}{y}.$$

To solve this equation let's put $\frac{x}{y} = t$, or $x = yt$, and instead of $\boldsymbol{U} : t$, let's write T, in such a way that T is a given function of t, and because of $\partial x = t\partial y + y\partial t$ our equation will be $\partial.\frac{v}{y^n} = \frac{p\partial t}{y^{n-1}} - Ty^{\lambda-n-1}\partial y$.

Let's now integrate, as permitted, and since

$$\int Ty^{\lambda-n-1}\partial y = \frac{y^{\lambda-n}}{\lambda - n}T - \int \frac{y^{\lambda-n}}{\lambda-n}T'\partial t,$$

assuming that $\partial T = T'\partial T$ we will have after integrating

$$\frac{v}{y^n} = -\frac{y^{\lambda-n}}{\lambda-n}T + \int \partial t(\frac{p}{y^{n-1}} + \frac{y^{\lambda-n}}{\lambda-n}T),$$

from which one sees that the formula $\frac{p}{y^{n-1}} + \frac{y^{\lambda-n}}{\lambda-n}T$ must be any function of t, that we will denote $\boldsymbol{B} : t$, and thus we will have this integrable equation :

$$\frac{v}{y^n} = \boldsymbol{B} : t - \frac{y^{\lambda-n}}{\lambda-n}T,$$

and therefore $v = y^n \boldsymbol{B} : t - \frac{y^\lambda}{\lambda - n} T$.

Let's put back now the value $\frac{x}{y}$ for t and $\boldsymbol{U} : t$ for T, where one must notice that the symbol $\boldsymbol{U}$ denote a given function of $\frac{x}{y}$, as it appears already in the given differential equation. But the symbol $\boldsymbol{U}$ will indicate here any arbitrary function of $\frac{x}{y}$, which is introduced in the ordinary integrations. Consequently we will have for the solution of our problem the following value of the function v, namely

$$v = y^n \boldsymbol{B} : \frac{x}{y} - \frac{y^\lambda}{\lambda - n} \boldsymbol{U} : \frac{x}{y}.$$

Here one will perhaps ask what will be the value of v, in the case where the exponent λ would be equal to n, since then the last member of our equation would become infinite? To rule out this difficulty let's put $\lambda = n + \omega$, where ω denotes an infinitely small quantity, and we will have

$$y^\lambda = y^n . y^\omega = y^n (1 + \omega l y),$$

we will have then $v = y^n \boldsymbol{B} : \frac{x}{y} - \frac{y^n(1+\omega l y)}{\omega} \boldsymbol{U} : \frac{x}{y}$. Now as $\boldsymbol{B}$ denotes an arbitrary function, it will be permitted to put instead of $\boldsymbol{B} : \frac{x}{y}$ this formula: $\frac{I}{\omega} \boldsymbol{U} : \frac{x}{y} + \boldsymbol{C} : \frac{x}{y}$, where $\boldsymbol{C}$ stands for any arbitrary function, and by substituting these values the infinite members cancel out and the integral needed for the case $\lambda = n$ will be $v = y^n \boldsymbol{C} : \frac{x}{y} - y^n l y \boldsymbol{U} : \frac{x}{y}$. Now we will be therefore able to solve the following problem.

1 PROBLEM

Find the complete integral of the differential equation

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0,$$

or figure out the nature of the function z.

Here we therefore have $P = 0$, and the first preliminary problem will first provide us with the integral needed, since we just have to write z instead of v, and from there our complete integral will be $z = \boldsymbol{U} : \frac{x}{y}$. Or we will take z to be an arbitrary homogeneous function of zero dimension of x and y.

Taking for example

$$z = \frac{xx - yy}{xx + yy},$$

we will have

$$\frac{\partial z}{\partial x} = \frac{4xyy}{(xx + yy)^2}$$

and

$$\frac{\partial z}{\partial y} = \frac{-4xxy}{(xx + yy)^2}$$

, thus

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$$

Similarly, by takind

$$z=\frac{x+y}{\sqrt{xx+yy}},$$

we will have

$$\frac{\partial z}{\partial x}=\frac{yy-xy}{(xx+yy)^3},$$

and

$$\frac{\partial z}{\partial y}=\frac{xx-xy}{(xx+yy)^3},$$

and from there we openly have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$$

2 PROBLEM

Find the complete integral of the differential equation of second degree:

$$xx\frac{\partial\partial z}{\partial x\partial y}+2xy\frac{\partial\partial z}{\partial x\partial y}+yy\frac{\partial\partial z}{\partial y^2}=0.$$

SOLUTION

One supposes therefore here that $Q=0$, and from there, since we proved above

$$Q=x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}-P,$$

we will solve this first degree differential equation

$$x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}=P$$

for which the integral is given by the second preliminary problem by putting P instead of v and $n=I$, thus we have $P=y\boldsymbol{U}:\frac{x}{y}$, where $\boldsymbol{C}$ is an arbitrary function. Let's put now instead of its value P and we will have to solve this first degree differential equation:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=y\boldsymbol{U}:\frac{x}{y}.$$

This equation being compared with the third preliminary problem gives us $n = 0$, $\lambda = I \quad and \quad v = z$; consequently the complete integral needed of our equation will be

$$z = \boldsymbol{B} : \frac{x}{y} - y\boldsymbol{U} : \frac{x}{y},$$

or, both functions are arbitrary, one will be able to put

$$z = \boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y},$$

which consequently has two arbitrary functions, as the nature of second order differential equations requires it.

3 Problem

Find the complete integral of this third order differential equation:

$$x^3\frac{\partial^3 z}{\partial x^3} + 3xxy\frac{\partial^3 z}{\partial x^2 \partial y} + 3xyy\frac{\partial^3 z}{\partial x \partial y^2} + y^3\frac{\partial^3 z}{\partial y^2} = 0.$$

SOLUTION

Here the point is therefore to take R to 0, and giving R its value above, we will have to solve this equation:

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} - 2Q = 0$$

which ,being compared with the one on the second preliminary problem, gives $v = Q \quad and \quad n = 2$, thus its complete integral is $Q = y^2\boldsymbol{U} : \frac{x}{y}$.

Now having $Q = x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P$, we will have to solve the equation

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P = y^2\boldsymbol{U} : \frac{x}{y}$$

which, being compared to the one on the third preliminary problem, gives $v = P, \quad n = 1, \quad \lambda = 2$, which being substituted gives the following integral

$$P = y\boldsymbol{B} : \frac{x}{y} - y^2\boldsymbol{U} : \frac{x}{y}$$

or since the functions are arbitrary, we will have

$$P = y^2\boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y}.$$

Finally therefore since $P = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$, we will have to solve this equation:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y^2\,\boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y},$$

which being compared to to the one on the third preliminary problem gives us $v = z,\;\; n = 0$, and for λ we will have two different values, either $\lambda = 2$ or $\lambda = 1$; as it is obvious that both can be treated similarly; consequently the complete integral of the problem will be

$$z = \boldsymbol{B} : \frac{x}{y} - y\,\boldsymbol{U} : \frac{x}{y} - y^2\,\boldsymbol{U} : \frac{x}{y},$$

or by changing the characters,signs of arbitrary functions, we have

$$z = \boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y} + y^2\,\boldsymbol{C} : \frac{x}{y}.$$

4 PROBLEM

Find te complete integral of this differential equation of fourth degree:

$$0 = x^4\frac{\partial^4 z}{\partial x^4} + 4xy^3\frac{\partial^4 z}{\partial x^3 \partial y} + 6xxyy\frac{\partial^4 z}{\partial x^2 \partial y^2} + 4xy^3\frac{\partial^4 z}{\partial x \partial y^3} + y^4\frac{\partial^4 z}{\partial y^4}.$$

SOLUTION

We will therefore have here $S = 0$, or

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} - 3R = 0,$$

which is compared to the second preliminary problem, gives $v = R$ and $n = 3$ and thus $R = y^3\,\boldsymbol{U} : \frac{x}{y}$. Putting instead of R its value, we will have to solve the equation

$$x\frac{\partial Q}{\partial y} + y\frac{\partial Q}{\partial y} - 3Q = y^3\,\boldsymbol{U} : \frac{x}{y},$$

which compared to the third preliminary, because of $v = Q,\;\; n = 2$ and $\lambda = 3$, gives

$$Q = y^2\boldsymbol{B} : \frac{x}{y} + y^3\,\boldsymbol{U} : \frac{x}{y},$$

or

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P = y^2\boldsymbol{B} : \frac{x}{y} + y^3\,\boldsymbol{U} : \frac{x}{y};$$

there is therefore $v = P,\ n = 1$ and $\lambda = 2$ or 3, thus we get

$$P = y\boldsymbol{B} : \frac{x}{y} + y^2\,\boldsymbol{U} : \frac{x}{y} + y^3\,\boldsymbol{U} : \frac{x}{y},$$

or

$$x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = y\,\boldsymbol{U} : \frac{x}{y} + y^2\boldsymbol{B} : \frac{x}{y} + y^3\,\boldsymbol{C} : \frac{x}{y},$$

which after comparing gives $v = z,\ n = 0$ and $\lambda = 1$ or 2, or 3, which gives

$$x = \boldsymbol{B} : \frac{x}{y} - y\,\boldsymbol{U} : \frac{x}{y} - y^2\,\boldsymbol{U} : \frac{x}{y} - y^3\,\boldsymbol{U} : \frac{x}{y},$$

or

$$z = \boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y} + y^2\,\boldsymbol{C} : \frac{x}{y} + y^3\boldsymbol{D} : \frac{x}{y}.$$

5 GENERAL PROBLEM

Find the complete integral of this differential equation of degree n:

$$x^n\frac{\partial nz}{\partial x^n} + \frac{n}{1}x^{n-1}y\frac{\partial^n z}{\partial x^{n-1}\partial y} + \frac{n}{1}\frac{n-1}{2}x^{n-2}y^2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ \ etc.$$

SOLUTION

Here it is easy to see that doing the successively the operations as in the previous problems, one will get to this complete integral

$$z = \boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y} + y^2\,\boldsymbol{C} : \frac{x}{y} + \ldots + y^{n-1}\boldsymbol{N} : \frac{x}{y},$$

where the number of arbitrary functions is n, and thus equal to the degree of the given equation; thus one sees that the integral of each degree has all the integrals of all the lowest degrees, moreover it has a term that belongs exclusively to the degree given.

Here are therefore the integrations of all the differential equations

1. $P = 0$
2. $Q = 0$
3. $R = 0$
4. $S = 0$ and so on.

by assigning to each of these letters the values that was given to them at the beginning, and the method that we used requires for each case as many integrations as the degree of the differential given. However, a young geometer, doing the previous calculations, has observed that all these solutions could be executed more easily upon one integration, and this method still has this big advantage on the one that we used so far, that it extends also to the integration of the composed differential equations with this general form:

$$Az + BP + CQ + DR + ES + \ etc \ = 0,$$

where all the degrees of the differentials are joined together, and where all the constant coefficients A, B, C, D, *etc.* can be taken arbitrarily. And the resolution of all these cases can always be done from the only second preliminary problem, which gives for the differential equation

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - nv = 0,$$

this complete integral $v = y^n \boldsymbol{U} : \frac{x}{y}$. To clarify this new method, we will add the following problems.

PROBLEM I

Find the complete integral of this differential equation of first degree $Az + BP = 0$, or

$$Az + B(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = 0.$$

SOLUTION

For this effect, let's put in the preliminary problem $v = az$, to have this equation:

$$a(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) - naz = 0,$$

so the integral is

$$z =:= y^n \boldsymbol{U} : \frac{x}{y}.$$

Now instead of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$, let's put its assigned value P, and the equation that we just integrated will be $aP - naz = 0$, which, compared with the given one $Az + BP$, gives $A = -na$ and $B = a$, consequently, $a = B$ and $A = -na$, or $A + nB = 0$. From this equation, we get the value of $n = -\frac{A}{B}$, the integral of the given equation will be $y^n \boldsymbol{U} : \frac{x}{y}$. This solution doesn't have anything that could not be done by the previous method, but the following problem will raise up the price of the new method.

PROBLEM II

Find the complete integral of this differential equation of second degree

$$Az + BP + CQ = 0.$$

SOLUTION

To solve this equation, let's assume that the preliminary problem $v = az + bP$, to have this integral

$$az + bP = y^n \boldsymbol{U} : \frac{x}{y},$$

which is consistent with the equation

$$a(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) - naz + b(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}) - nbP = 0.$$

Let's put now in this equation, instead of $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}$, its absolute value got from the formulas assumed at the beginning, which is P, instead of the formula $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}$ let's put this absolute value $Q + P$, and we will have this equation:

$$aP + bQ + bP - naz - nbP = 0,$$

or

$$-naz + (a + b - nb)P + bQ = 0,$$

which being compared with the given form

$$Az + BP + CQ = 0,$$

gives us for the letters a and b the following values: $b = C$, $a = B - C + nC$, and $0 = A + n(n-1)C$, from where one must get the value of n.

But since this last equation is of second degree, it will have two roots, namely α and β, from each one, we have particular values for a and b, which are:

$$a = B + (\alpha - 1)C, \quad b = C \quad for\ n = \alpha$$

$$a = B + (\beta - 1)C, \ b = C \quad for\ n = \beta.$$

From there, we will have two integral equations:

$$(B + (\alpha - 1)C))z + CP = y^\alpha \boldsymbol{U} : \frac{x}{y}$$

$$(B + (\beta - 1)C)z + CP = y^\beta \boldsymbol{B} : \frac{x}{y}$$

Now from these two equations, we just have to chase the letter P, which we do by taking the difference, we get $(z - \beta)Cz = y^\alpha \boldsymbol{U} : \frac{x}{y} - y^\beta \boldsymbol{B} : \frac{x}{y}$; and since the functions are absolutely arbitrary, one will be able to represent the integral in this form: $z = y^\alpha \boldsymbol{U} : \frac{x}{y} - y^\beta \boldsymbol{B} : \frac{x}{y}$.

Corollary

From there we easily deduce the integral of the equation $Q = 0$, that we solved above; we just have to suppose $A = 0$ and $B = 0$ and $C = 1$ and then the equation for the number n becomes $n(n-1) = 0$, so the roots are $n = 0$ and $n = 1$, consequently $\alpha = 0$ and $\beta = 1$, so the integral of this case will be

$$z = \boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y}.$$

PROBLEM II

Find the complete integral of the differential equation of third degree:

$$Az + BP + CQ + DR = 0,$$

SOLUTION

To get to the solution of this problem, suppose in the second preliminary problem that

$$v = aZ + bP + cQ,$$

and the integration gives us first this equation:

$$aZ + bP + cQ = y^n \boldsymbol{U} : \frac{x}{y},$$

and this integral is convenient with the following differential equation:

$$a(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) - naZ + b(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}) - nbP + c(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}) - ncQ = 0$$

Now instead of the differential formulas, let's put their finite values, and we will get to this equation:

$$aP + b(P + Q) + c(R + 2Q) - naZ - nbP - ncQ = 0$$

which is reduced to this form:

$$-naZ + (a + b(1-n)P) + (b + (2-n)c)Q + cR = 0,$$

which being compared with the proposed gives us the equations of the following conditions:

$$A = -na; \quad B = a + b(1-n); \quad C = b + (2-n)c; \quad D = c.$$

Having therefore from the last $D = C$, the third one will give us $b = C + (n-2)D$; next the second equation gives us $a = B + (n-1)C + (n-1)(n-2)D = 0$; and this value substituted in the first gives this final value:

$$A + nB + n(n-1)C + n(n-1)(n-2)D = 0;$$

which being of third degree has three roots, namely $\alpha,\ \beta\ \gamma$. each of these roots will give us 2 particular values for the letters $a,\ b,\ c$ which being reported to the root α, suppose that for the root β, we have $a',\ b'\ c'$ and for the root γ we have these ones: $a'',\ b'', c''$, each of these case will therefore give us a particular integral equation, and these equations will be

$$aZ + bP + cQ = y^{\alpha} b\boldsymbol{U} : \frac{x}{y}$$
$$a'Z + b'P + c'Q = y^{\beta}\boldsymbol{B} : \frac{x}{y}$$
$$a''Z + b''P + c''Q = y^{\gamma}\boldsymbol{C} : \frac{x}{y}.$$

Now it is easy to find for each of these equations some multiplicands such that, adding the product together the quantities P and Q are destroyed and since these multiplicands don't change the nature of the arbitrary functions, one will get this way to the final equation

$$z = y^{\alpha}\boldsymbol{U} : \frac{x}{y} + y^{\beta}\boldsymbol{B} : \frac{x}{y} + y^{\gamma}\boldsymbol{C} : \frac{x}{y}$$

which express the complete integral of our differential equation proposed.

Corollary

To get from there the integral of the equation $R = 0$, we just have to put $A = 0,\ B = 0,\ C = 1$, and then the cubic equation for the number n will become $n(n-1)(n-2) = 0$, so the three roots are clearly $0, 1, 2$ so that $\alpha = 0,\ \beta = 1,\ \gamma = 2$; where we get the integral needed for this case

$$\boldsymbol{U} : \frac{x}{y} + y\boldsymbol{B} : \frac{x}{y} + y^2\boldsymbol{C} : \frac{x}{y},$$

which matches perfectly with the one found above.

PROBLEM III

Find the complete integral of this differential equation of fourth degree:

$$Az + BP + CQ + DR + ES = 0.$$

SOLUTION

To solve this equation, let's put in the preliminary second problem $v = az + bP + cQ + dR$, and we will have first this integral:

$$az + bP + cQ + dR = y^n \boldsymbol{U} : \frac{x}{y},$$

which will be convenient with this differential equation:

$$a(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) - naZ + b(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}) - nbP + c(x\frac{\partial Q}{\partial x} +$$

$$y\frac{\partial Q}{\partial y}) - ncQ + d(x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y}) - n\partial R = 0$$

Now instead of writing the differential formulas, we write their finite values, to get this equation:

$$aP - naz + b(Q + P) - nbP + c(R + 2Q) - ncQ + \partial(S + 3R) - n\partial R = 0$$

for which rearranging terms will give

$$-naz + (a + b(1 - n))P + (b + c(2 - n))Q + (c + d(3 - n))R + dS = 0.$$

It just remains to identity this form with the proposed one, which yields the following five equalities:

1. $A = -na$;
2. $B = a + (1 - n)b$
3. $C = b + (2 - n)c$
4. $D = c + (3 - n)d$
5. $E = d$.

The last gives us first $d = E$; the fourth gives $c = D + (n - 3)E$; then form the third , we get $b = C(n - 2)D + (n - 2)(n - 3)E$; the second will give $a = B + (n - 1)C + (n - 1)(n - 2)D + (n - 1)(n - 2)(n - 3)E$; finally the first takes us to this equation for the determination of the number n :

$$A + nB + n(n - 1)C + (n - 1)(n - 2)D + n(n - 1)(n - 2)(n - 3)E = 0.$$

This last equation being of fourth degree, let $\alpha,\ \beta,\ \gamma\ \delta$ the fourth values of the number n, so each will produce for the letters a, b, c, d, particular values. Let's put for the root α these same letters a, b, c, d, for the root β : a', b', c', d' for γ : a'', b'', c'', d'', and for δ : a''', b''', c''', d'''; and so we will have four different forms of the integral equation found , namely:

$$aZ + bP + cQ + dR = y^\alpha \boldsymbol{U} : \frac{x}{y}$$

$$a'Z + b'P + c'Q + d'R = y^\beta \boldsymbol{B} : \frac{x}{y}$$

$$a''Z + b''P + c''Q + d''R = y^\gamma \boldsymbol{C} : \frac{x}{y}$$

$$a'''Z + b'''P + c'''Q + d'''R = y^\delta \boldsymbol{D} : \frac{x}{y}.$$

After finding these four equations, it is easy to eliminate the three quantities P, Q, R, in such a way that it will remain on the left just the the quantity Z, and as the functions from the left, being multiplied by certain constants, don't change their nature, we will get the final equation:

$$y^\alpha \boldsymbol{U} : \frac{x}{y} + y^\beta \boldsymbol{B} : \frac{x}{y} + y^\gamma \boldsymbol{C} : \frac{x}{y} + y^\delta \boldsymbol{D} : \frac{x}{y};$$

where it is good to point out that to find this equation we didn't need to find the values of a, b, c, d, neither even the multiplicands, for the elimination of the quantities: P, Q, R.

GENERAL PROBLEM IV

Find the complete integral of the differential equation of any degree:

$$AZ + BP + CQ + DR + \ldots = 0$$

SOLUTION

All the solution of this equation reduces to the equation to determine all the values of the number n; and it is clear from the previous problems that this equation will have the form

$$A + nB + n(n-1)C + n(n-1)(n-2)D + \ldots = 0,$$

which will go up to the same degree of the proposed differential equation, and from there the number n will have as many values that we denote by the letters $\alpha, \beta, \gamma, \delta, \ldots$ and then the complete integral of the proposed equation will be

$$y^\alpha \boldsymbol{U} : \frac{x}{y} + y^\beta \boldsymbol{B} : \frac{x}{y} + y^\gamma \boldsymbol{C} : \frac{x}{y} + y^\delta \boldsymbol{D} : \frac{x}{y} + \ldots$$

which has as many arbitrary functions as the given differential.

Here it is good to notice that the since the two variables x, y are used in the calculations, instead of the powers $y^\alpha, y^\beta, y^\gamma, \dots$ we will be able to put also the similar powers of x, namely $x^\alpha, x^\beta, x^\gamma, \dots$. And in fact, if we consider the formula $y^\alpha \boldsymbol{U} : \frac{x}{y}$, since $\frac{x^\lambda}{y^\lambda}$ is also a function of $\frac{x}{y}$, instead of $\boldsymbol{U} : \frac{x}{y}$ we will be able to put $\frac{x^\lambda}{y^\lambda}\boldsymbol{F} : \frac{x}{y}$; and so we will have $x^\lambda y^{\alpha-\lambda}\boldsymbol{F} : \frac{x}{y}$. So taking $\lambda = \alpha$, instead of the formula $y^\alpha \boldsymbol{U} : \frac{x}{y}$ we will be able to put $x^\alpha \boldsymbol{U} : \frac{x}{y}$; and it is also clear that we will be able to write in general $x^\mu y^\nu \boldsymbol{U} : \frac{x}{y}$, as long as the sum of the exponents μ and ν are equal to α i.e., $\mu + \nu = \alpha$.

This solution will have no difficulties, as long as the values of the exponents n, that we suppose to be $\alpha, \beta, \gamma, \delta, \dots$ are all reals and different to each other. But in the case where some of these values are imaginary or equal to each other, one turns to certain reductions to make the integral real in the first case; or for the other case the necessary number of arbitrary functions stays non-reduced, without which the integral will no longer be complete.

To overcome all these difficulties, let's start by considering the two cases where the values of n are found imaginaries, namely α, β, and one knows that these two values always get written to the forms $\alpha = \mu + \nu V - 1$ and $\beta = \mu - \nu V - 1$, and so the terms of the integral which depend on those values will be $y^{\mu+\nu V-1}\boldsymbol{U} : \frac{x}{y}$ and $y^{\mu-\nu V-1}\boldsymbol{B} : \frac{x}{y}$; and to reduce them to the reality ,assume that

$$\boldsymbol{U} : \frac{x}{y} = \boldsymbol{F} : \frac{x}{y} + \boldsymbol{O} : \frac{x}{y}$$

and

$$\boldsymbol{B} : \frac{x}{y} = \boldsymbol{F} : \frac{x}{y} - \boldsymbol{O} : \frac{x}{y},$$

and then these two terms in turns will be reduced to the form:

$$y^\mu \boldsymbol{F} : \frac{x}{y}(y^{\nu V-1} + y^{-\nu V-1}) + y^\mu (y^{\nu V-1} - y^{-\nu V-1})\boldsymbol{O} : \frac{x}{y}.$$

Let's put here in the imaginary powers e^{ly} instead of y, taking e to be the number for which the hyperbolic logarithm is 1; and the first formula $y^{\nu V-1} + y^{-\nu V-1}$ will become $e^{\nu V-1ly} + e^{-\nu V-1ly}$,and the other $y^{\nu V-1} - y^{-\nu V-1}$ will become $e^{\nu V-1ly} - e^{-\nu V-1ly}$. However, we know by the reductions known that

$$e^{vV-1} + e^{-vV-1} = 2\cos v,$$

and

$$e^{vV-1} - e^{-vV-1} = 2V - 1fin\ v.$$

So as $v = \nu ly$, our two terms will have the form

$$y^\mu .2\cos v.\boldsymbol{F} : \frac{x}{y} + y^\mu .2V - 1fin\ v.\boldsymbol{O} : \frac{x}{y};$$

where we can discard the real constant coefficients and the imaginaries. We will then have, instead of the two terms proposed, these ones:

$$y^{\mu}\cos vly\boldsymbol{F}:\frac{x}{y}+y^{\mu}fin\ vly.\boldsymbol{O}:\frac{x}{y};$$

all the times that $\alpha=\mu+\nu V-1$ and $\beta=\mu-\nu V-1$. from there it is clear that when the number of imaginary values of n is $4,6,8,10,\ldots$ since each couple always get reduced to these two formulas $\mu+\nu V-1$ and $\mu-\nu V-1$, the reduction will be always done the same way.

To give an example, let's take the case where the equation, to determine the number n becomes $1+nn=0$, which is of second degree, where we had found $A+nB+n(n-1)C=0$, one must take A=B=C=1, in such a way that the differential equation to be solved will be in this case $z+P+Q=0$, or expanding it:

$$z+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+xx\frac{\partial\partial z}{\partial x^2}+2xy\frac{\partial\partial z}{\partial x\partial y}+yy\frac{\partial\partial z}{\partial y^2}=0.$$

And since for n we have the values $\alpha=V-1,\ \beta=-V-1$, and thus $\mu=0\ and\ \nu=1$, we conclude first that $z=cos.ly\boldsymbol{F}:\frac{x}{y}+sin.ly.\boldsymbol{O}:\frac{x}{y}$. To better clarify these case, let's $\boldsymbol{F}:\frac{x}{y}=0\ and\ \boldsymbol{O}:\frac{x}{y}=\frac{x}{y}$, in such a way that a particular integral will be $z=\frac{x}{y}sin.ly$, where we get $\frac{\partial z}{\partial x}=\frac{x}{y}.sin.ly$ and $\frac{\partial z}{\partial y}=-\frac{x}{yy}.sin.ly+\frac{x}{yy}cos.ly$, and next $\frac{\partial\partial z}{\partial x^2}=0$ and

$$\frac{\partial\partial z}{\partial x\partial y}=-\frac{x}{yy}sin.ly+\frac{x}{yy}cos.ly\ \ and\ \ \frac{\partial\partial z}{\partial y^2}=\frac{x}{y^3}sin.ly-\frac{3x}{y^3}cos.ly.$$

These values being substituted in the equation $z+P+Q=0$, will give

$$z=\frac{x}{y}cos.ly\quad P=\frac{x}{y}cos.ly\quad Q=-\frac{x}{y}sin.ly-\frac{x}{y}cos.ly$$

for which the sum is $z+P+Q=0$

Let's move to the case where one or two values of of n become equal to each other. Assume first that $\alpha=\beta$, and in the integral form found the two first terms $y^{\alpha}\boldsymbol{U}:\frac{x}{y}+y^{\beta}\boldsymbol{B}:\frac{x}{y}$ will be reduced to only one function, and thus the integral will no longer be complete. To fill this number, let's $\beta=\alpha+\omega$, taking ω to be infinitely small, and because of $y^{\beta}=y^{\alpha}.y^{\omega}$ and $y^{\omega}=1+\omega ly$, we will have $y^{\beta}=y^{\alpha}+\omega y^{\alpha}ly$, where the two first terms will become $y^{\alpha}\boldsymbol{U}:\frac{x}{y}+y^{\alpha}\boldsymbol{B}:\frac{x}{y}+\omega y^{\alpha}ly\boldsymbol{B}:\frac{x}{y}$ instead of the two first terms,one can simply write $y^{\alpha}\boldsymbol{U}:\frac{x}{y}$, and $\boldsymbol{B}:\frac{x}{y}$ instead of $y^{\omega}\boldsymbol{B}:\frac{x}{y}$; in such a way that instead of the first two terms, we will have : $y^{\alpha}\boldsymbol{U}:\frac{x}{y}+y^{\alpha}ly\boldsymbol{B}:\frac{x}{y}$.

To give an example of this case, assume that the equation to determine the number n, is $nn=0$, and this equation will be of second degree, for which we have in general

$A+nB+n(n-1)C=0$, where we must put $A=0,\ B+1,\ C=1$, in such a way that the differential equation to be integrated will be $P+Q=0$, or

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+xx\frac{\partial\partial z}{\partial x^2}+2xy\frac{\partial\partial z}{\partial x\partial y}+yy\frac{\partial\partial z}{\partial y^2}=0.$$

Having therefore for the resolution of this equation $nn=0$, the two equal values of n will be $\alpha=0$ *and* $\beta=0$; consequently the complete integral needed of this equation is $z=\boldsymbol{U}:\frac{x}{y}+y\boldsymbol{B}:\frac{x}{y}$; where it will be useful to show how this value satisfy in general the proposed equation. To do this, we will differentiate these formulas according to the rules established below $\partial\boldsymbol{U}:v=\partial v\,\boldsymbol{U'}:v$ and $\partial\boldsymbol{U'}:v=\partial v\,\boldsymbol{U''}:v$, and we will find:

$$\frac{\partial z}{\partial x}=\frac{1}{y}\boldsymbol{U'}:\frac{x}{y}+\frac{ly}{y}\boldsymbol{B'}:\frac{x}{y};$$

$$\frac{\partial z}{\partial y}=-\frac{x}{yy}\boldsymbol{U'}:\frac{x}{y}+\frac{1}{y}\boldsymbol{B}:\frac{x}{y}-\frac{xly}{yy}\boldsymbol{B'}:\frac{x}{y};$$

$$\frac{\partial\partial z}{\partial x^2}=\frac{1}{yy}\boldsymbol{U'}:\frac{x}{y}+\frac{1}{y}\boldsymbol{B''}:\frac{x}{y};$$

$$\frac{\partial\partial z}{\partial x\partial y}=-\frac{1}{yy}\boldsymbol{U'}:\frac{x}{y}-\frac{x}{y^3}\boldsymbol{U''}:\frac{x}{y}+\frac{1}{yy}\boldsymbol{B'}:\frac{x}{y}-\frac{ly}{yy}\boldsymbol{B'}:\frac{x}{y}-\frac{xly}{y^3}\boldsymbol{B''}:\frac{x}{y};$$

$$\frac{\partial\partial z}{\partial y^2}=\frac{2x}{y^3}\boldsymbol{U'}:\frac{x}{y}+\frac{xx}{y^4}\boldsymbol{U''}:\frac{x}{y}-\frac{1}{yy}\boldsymbol{B}:\frac{x}{y}-\frac{2x}{y^3}\boldsymbol{B'}:\frac{x}{y}+\frac{2xly}{y^3}\boldsymbol{B'}:\frac{x}{y}+\frac{x^2ly}{y^4}\boldsymbol{B''}:\frac{x}{y};$$

from where we get the following formula:

$$P=\frac{x}{y}\boldsymbol{U'}:\frac{x}{y}+\frac{xly}{y}\boldsymbol{B'}:\frac{x}{y}+\boldsymbol{B}:\frac{x}{y}-\frac{x}{y}\boldsymbol{U'}:\frac{x}{y}-\frac{xly}{y}\boldsymbol{B'}:\frac{x}{y};$$

or $P=-\boldsymbol{B}:\frac{x}{y}$. The same way , we will find $Q=-\boldsymbol{B}:\frac{x}{y}$. Thus we get directly that $P+Q=0$. This seems to be so necessary that one finds nowhere particular rules to differentiate the functions with two variables.

Considering now also the case where not only that the two roots $\alpha=\beta$ but there is a third one γ which is equal to them. For the two first $\alpha=\beta$, we just reduce their corresponding terms to this form: $y^\alpha\boldsymbol{U}:\frac{x}{y}+y^\alpha ly\boldsymbol{B}:\frac{x}{y}$ on which we must add the third term $y^\gamma\boldsymbol{C}:\frac{x}{y}$, to join the first one. Now let's put $\gamma=\alpha+\omega$, and since $y^\omega=1+\omega ly+\frac{1}{2}\omega^2(ly^2)$, we must go here up to the third term, since the second term will join the second of the preceding terms. from there it is clear that these three terms, by changing the arbitrary functions, will be reduced to the following three terms:

$$y^\alpha\boldsymbol{U}:\frac{x}{y}+y^\alpha ly\boldsymbol{B}:\frac{x}{y}+y^\alpha(ly)^2\boldsymbol{C}:\frac{x}{y}.$$

To give an example, consider the case where the equation for the number n has this form: $1\ 3n + 3nn\ n^3 = 0$, for which the three roots are all equal to each other, namely $\alpha = \beta = \gamma = 1$. This case belongs therefore to the differential equation of third degree $Az + BP + CQ + DR = 0$, for which we had found: $A + nB + nnC + n^3D = 0 - nC - 3nnD + 2nD$ we must therefore do: $A = 1,\ B - C + 2D = -3,\ C - 3D = +3\ and\ D = -1$, and so $C = 0,\ B = -1\ and\ A = 1$, in such a way that our differential equation will be: $z - P + 0.Q - R = 0$, for which the complete integral , because of $\alpha = 1$, will be:

$$y\boldsymbol{U} : \frac{x}{y} + yly\boldsymbol{B} : \frac{x}{y} + y(yl)^2\boldsymbol{C} : \frac{x}{y}.$$

To clarify this by an example, let's do: $\boldsymbol{U} = 0,\ \boldsymbol{B} = 0\ and\ \boldsymbol{C} : \frac{x}{y} = \frac{x}{y}$; in such a way that a particular integral will be $x(ly)^2$, from where we have the following differentials:

$$\frac{\partial z}{\partial x} = (ly)^2;\ \frac{\partial z}{\partial y} = \frac{2xly}{y};$$

$$\frac{\partial\partial z}{\partial x^2} = 0;\ \frac{\partial\partial z}{\partial x\partial y} = \frac{2ly}{y};\ \frac{\partial\partial z}{\partial y^2} = \frac{2x}{yy} - \frac{2xly}{yy};$$
$$\frac{\partial^3 z}{\partial x^3} = 0;\ \frac{\partial^3 z}{\partial x^2\partial y} = 0;\ \frac{\partial^3 z}{\partial x\partial y^2} = \frac{2}{yy} - \frac{2ly}{yy};$$
$$\frac{\partial^3 z}{\partial y^3} = -\frac{4x}{y^3} - \frac{2x}{y^3} + \frac{4xly}{y^3} = -\frac{6x}{y^3} + \frac{4xly}{y^3}.$$

From there we get

$$z = x(ly)^2;\ P = x(ly)^2 + 2xly\ and\ R = -2xly,$$

from where we get: $z - P - R = 0$; which agrees perfectly.

From there it is already obvious that the number n had four equal values, namely $\alpha = \beta = \gamma = \delta$. Instead of the four terms that immediately get into the integral, we must put these ones:

$$z = y^\alpha\boldsymbol{U} : \frac{x}{y} + y^\alpha ly\boldsymbol{B} : \frac{x}{y} + y^\alpha(ly)^2\boldsymbol{C} : \frac{x}{y} + y^\alpha(ly)^3\boldsymbol{D} : \frac{x}{y} + \ldots.$$

And so whatever number of equal roots, the reduction of the integral will no longer have any difficulties. In the remaining one understands easily that the two letters x and y can be exchanged in all those formulas.

To prove that, I will show that instead of the terms: $y^\alpha\boldsymbol{U} : \frac{x}{y} + y^\alpha(ly)\boldsymbol{B} : \frac{x}{y}$ one will be able to write $x^\alpha\boldsymbol{U} : \frac{x}{y} + x^\alpha(lx)\boldsymbol{B} : \frac{x}{y}$. For this fact I notice that because both terms bears an arbitrary function of $\frac{x}{y}$, one will be able to multiply it by $\frac{x^\alpha}{y^\alpha}$, which gives

$x^\alpha \boldsymbol{U} : \frac{x}{y} + x^\alpha(ly)\boldsymbol{B} : \frac{x}{y}$; next since $l\frac{x}{y} = lx - ly$ is a function of $\frac{x}{y}$, instead of $\boldsymbol{U} : \frac{x}{y}$, we will be able to write $\boldsymbol{U} : \frac{x}{y} + l\frac{x}{y}\boldsymbol{B} : \frac{x}{y}$, and then we will have

$$x^\alpha \boldsymbol{U} : \frac{x}{y} + x^\alpha(lx)\boldsymbol{B} : \frac{x}{y};$$

from where one understands easily that this permutation can always happen.

The integration of this enough general differential equation:

$$Az + BP + CQ + DR + ER + ES + \ldots = 0,$$

where $P, Q, R, S, \ldots$ mark the differential formulas pointed out above, will be looked as an excellent part of this analysis that solve functions of two variables, and that one must distinguish from the ordinary analysis which works only single variable functions. As it is now clear that these two kinds of analysis are very essentially different to each other, not only with respect to the functions that they solve, but also with respect to the methods used to solve them. This is why the denomination of partial differences, which many geometers use, to mark the analysis of functions of two variables, doesn't seem much to me to express their true character.

Besides that difference, on can sometimes notice a good harmony between these two kinds of analysis. So when solving, in the ordinary analysis, this differential equation: $Az + Bx\frac{\partial z}{\partial x} + Cx^2\frac{\partial\partial z}{\partial x^2} + Dx^3\frac{\partial^3 z}{\partial x^3} + \ldots = 0$; and that we ask what function of x one must give to the quantity z, For this equation to be complete: The ordinary method of integration leads to this algebraic equation: $A + nB + n(n-1)C + n(n-1)(n-2)D + n(n-1)(n-2)(n-3)E + \ldots = 0$; from where one must get all the roots $\alpha, \beta, \gamma, \delta, \ldots$ of n, and the complete integral is expressed in the following way:

$$z = \boldsymbol{U}x^\alpha + \boldsymbol{B}x^\beta + \boldsymbol{C}x^\gamma + \boldsymbol{D}x^\delta + \ldots$$

where the letters: $\boldsymbol{U}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}, \ldots$. stand for arbitrary constants. this form has therefore a very good link with the form of the integral that we have found above for the function z of the two variables of x and y.