# EXPANSION OF THE INTEGRAL FORMULA OF $\int \frac{\partial z(3+zz)}{(1+zz)\sqrt[4]{(1+6zz+z^4)}}$ THROUGH LOGARITHMS AND CIRCULAR ARCS

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## 1. §1

This is not able to be done unless it is established that $z = \frac{1+x}{1-x}$. From here we will make

$$\partial z = \frac{2\partial x}{(1-x)^2} \text{ and } \frac{3+zz}{1+zz} = \frac{2(1-x+xx)}{1+xx} = \frac{2(1+x^3)}{(1+x)(1+xx)}.$$

Then certainly

$$1+6zz+z^4 = \frac{8(1+x^4)}{(1-x)^4}, \text{ and therefore}$$

$$\sqrt[4]{(1+6zz+z^4)} = 2^{\frac{3}{4}}\frac{\sqrt[4]{1+x^4}}{1-x}.$$

The proposed substituted formula then has this form:

$$2^{\frac{5}{4}}\int \frac{\partial x(1+x^3)}{(1-x^4)\sqrt[4]{1+x^4}}.$$

## 2. §2

We tear apart the aforementioned into a form which has two parts: $2^{\frac{5}{4}}(P+Q)$, so to make:

$$P = \int \frac{\partial x}{(1-x^4)\sqrt[4]{1+x^4}} \text{ and } Q = \int \frac{x^3\partial x}{(1-x^4)\sqrt[4]{1+x^4}},$$

which we unfold. For the first part we set up $\frac{x}{\sqrt[4]{(1+x^4)}} = t$, in order to have $P = \int \frac{t\partial x}{x(1-x^4)}$. The substitution becomes becomes $\frac{x^4}{1+x^4} = t^4$, and from here $x^4 = \frac{t^4}{1-t^4}$, and therefore $1-x^4 = \frac{1-2t^4}{1-t^4}$. Next, from $4\ln x = 4\ln t - \ln(1-t^4)$, we have that $\frac{\partial x}{x} = \frac{\partial t}{t(1-t^4)}$, these give

$$P = \int \frac{\partial t}{1-2t^4}.$$

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But for the other part, $Q$, place $1+x^4=u^4$, from which $\sqrt{(1+x^4)}=u$ [1] is made and $x^3\partial x=u^3\partial u$. It is then deduced that:

$$Q=\int\frac{uu\partial u}{2-u^4},$$

in such a way that we have transformed (the original) into rational functions .

## 3. §3

Now we are able to manage these convenient formulas. For the former we place $t=\frac{p}{\sqrt[4]{2}}$, and in this way will have $P=\frac{1}{\sqrt[4]{2}}\int\frac{\partial p}{1-p^4}$. Now it is true that

$$\frac{1}{1-p^4}=\frac{1}{2}\frac{1}{1-pp}+\frac{1}{2}\frac{1}{1+pp},$$

and

$$P=\frac{1}{2\sqrt[4]{2}}\int\frac{\partial p}{1-pp}+\frac{1}{2\sqrt[4]{2}}\int\frac{\partial p}{1+pp}.$$

This can be rewritten:

$$P=\frac{1}{4\sqrt[4]{2}}\ln\frac{1+p}{1-p}+\frac{1}{2\sqrt[4]{2}}\arctan p,$$

and the former part will be as follows:

$$2^{\frac{5}{4}}P=\frac{1}{2}\ln\frac{1+p}{1-p}+\arctan p.$$

We note that $p=t\sqrt[4]{2}$, and furthermore it is true that $t=\frac{x}{\sqrt[4]{1+x^4}}$. And since we placed $z=\frac{1+x}{1-x}$ we have $x=\frac{z-1}{z+1}$, so that all of the former part of the integral will be able to be expressed by means of $z$.

## 4. §4

For the other part $Q$, place

$$u=q\sqrt[4]{2}$$

and $Q$ becomes $\frac{1}{\sqrt[4]{2}}\int\frac{qq\partial q}{1-q^4}$. Now it is true that

$$\frac{qq}{1-q^4}=\frac{1}{2}\frac{1}{1-qq}-\frac{1}{2}\frac{1}{1+qq},$$

which becomes

$$\int\frac{qq\partial q}{1-q^4}=\frac{1}{2}\int\frac{\partial q}{1-qq}=\frac{1}{2}\int\frac{\partial q}{1+qq}=\frac{1}{2}\ln\frac{1+q}{1-q}-\frac{1}{2}\arctan q.$$

Therefore, this gives forth

$$Q=\frac{1}{4\sqrt[4]{2}}\ln\frac{1+q}{1-q}-\frac{1}{2\sqrt[4]{2}}\arctan q.$$

Consequently, the real other part of the integral will be

$$2^{\frac{5}{4}}Q=\frac{1}{2}\ln\frac{1+q}{1-q}-\arctan q.$$

We note that $q=\frac{u}{\sqrt[4]{2}}$, and furthermore it is true $u=\sqrt[4]{(1+x^4)}$. Finally we see $x=\frac{z-1}{z+1}$ is true.

[1] This should read $\sqrt[4]{1+x^4}=u$.

## 5. §5

Therefore, seeing that each of these values are known, the integral of the proposed formula will be

$$\int \frac{\partial z(3+zz)}{(1+zz)\sqrt[4]{1+6zz+z^4}} + {}^{2}\ \frac{1}{2}\ln\frac{1+p}{1-p} + \frac{1}{2}\ln\frac{1+q}{1-q} + \arctan p - \arctan q$$

whereby it should be noted that

$$p = \frac{z-1}{\sqrt[4]{1+6zz+z^4}} \text{ and } q = \sqrt[4]{1+6zz+z^4}.\ ^{3}$$

## 6. §6

Therefore, with the substitution of these values, our integral will be

$$\frac{1}{2}\ln\frac{\sqrt[4]{(1+6zz+z^4)}+z-1}{\sqrt[4]{(1+6zz+z^4)}-z+1} + \frac{1}{2}\ln\frac{1+\sqrt[4]{(1+6zz+z^4)}}{z-\sqrt[4]{(1+6zz+z^4)}}$$
$$+ \arctan\frac{1-z}{\sqrt[4]{(1+6zz+z^4)}} - \arctan\sqrt[4]{(1+6zz+z^4)}\ ^{4}$$

whereby it is noted both circular arcs can thus be combined as one in order to give

$$\arctan\frac{z-1-\sqrt{(1+6zz+z^4)}}{z\sqrt[4]{(1+6zz+z^4)}}.\ ^{5}$$

Moreover, both logarithms are able to be combined into one

$$\frac{1}{2}\ln\frac{z\sqrt[4]{(1+6zz+z^4)}+z-1+\sqrt{(1+6zz+z^4)}}{z\sqrt[4]{(1+6zz+z^4)}-z+1-\sqrt{(1+6zz+z^4)}}.\ ^{6}$$

## 7. §7

These, though convenient, can be simplified. If in fact we place esteem on brevity and let $\sqrt[4]{(1+6zz+z^4)} = v$, the logarithmic part of our integral will be

$$\frac{1}{2}\ln\frac{vz+z-1+vv}{vz-z+1-vv} = \frac{1}{2}\ln\frac{(1+v)(z-1+v)}{(v-1)(z-1-v)}\ ^{7}$$

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[2] This + should be =

[3] Here (and following), $q$ should be $\frac{\sqrt[4]{1+6zz+z^4}}{z+1}$. We see this since $q = \frac{u}{\sqrt[4]{2}}$, $u = \sqrt[4]{(1+x^4)}$ and $x = \frac{z-1}{z+1}$.

[5] This should read

$$\arctan\frac{z-1-\sqrt{(1+6zz+z^4)}}{2z\sqrt[4]{(1+6zz+z^4)}}$$

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[6] This equation should read

$$\frac{1}{2}\ln\frac{2z\sqrt[4]{(1+6zz+z^4)}+z^2-1+\sqrt{(1+6zz+z^4)}}{2z\sqrt[4]{(1+6zz+z^4)}-z^2+1-\sqrt{(1+6zz+z^4)}}$$

[7] This equation should read

$$\frac{1}{2}\ln\frac{2vz+zz-1+vv}{2vz-zz+1-vv} = \frac{1}{2}\ln\frac{(v+z)^2-1}{1-(v-z)^2}$$

the other circular part is

$$\arctan \frac{1-z-vv}{vz}.$$ [8]

[8] This equation should read

$$\arctan \frac{zz-1-vv}{2vz}.$$