An Analytic Exercise

Where in particular a most general summation of a series is given

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Translated from the Latin by Walter Jacob

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§. 1. Having recently revealed that the nature of curves can very adequately be expressed by the relationship between the arc length of the curve itself and its amplitude, it occurred to me that I might apply this same reasoning to the equilateral hyperbola, which may be expressed by the equation $yy = xx - 1$. Therefore let CB be the axis of this hyperbola, with point C its center and A its vertex, and let $CA = 1$ be the semiaxis. Now from the point M of the curve, draw the ordinate MP, so that $CP = x$ is the abscissa and $PM = y$ the ordinate, and that $y = \sqrt{xx - 1}$. From now on let the line CV be the asymptote of this hyperbola, with the axis establishing an angle with the semirectum, from which an ordinate from PM to the asymptote at S is induced at every point, so that $PS = CP = x$ and $CS = x\sqrt{2}$.

§. 2. Now, further, let $AM = s$ be the arc of the hyperbola, draw the normal to the curve MN, and let the angle $ANM = \phi$ be called the amplitude of the arc AM. Now since $y\partial y = x\partial x$, $PN = x = CP$ is the subnormal, and from this we see that $\tan\phi = \frac{y}{x} = \frac{\sqrt{xx-1}}{x}$, from which the amplitude may be expressed by the quantity x, so that $xx = \frac{\cos^2\phi}{\cos^2\phi - \sin^2\phi}$. And so $x = \frac{cos\phi}{\sqrt{cos2\phi}}$, from which it is deduced that $\partial x = \frac{\partial\phi(\cos\phi\sin 2\phi - \sin\phi\cos 2\phi)}{\cos 2\phi\sqrt{\cos 2\phi}}$. Therefore since $\frac{\partial x}{\partial s} = \sin\phi$ then $\partial s = \frac{\partial x}{\sin\phi}$, and so $\partial s = \frac{\partial\phi}{\cos 2\phi\sqrt{\cos 2\phi}}$, which is the differential equation between an arc of the curve s and its amplitude ϕ.

[1] *Translator's note:* The original paper includes typesetting instruction for a figure, but no figure is included in the publication. The figure included here is an attempted recreation of the missing illustration.

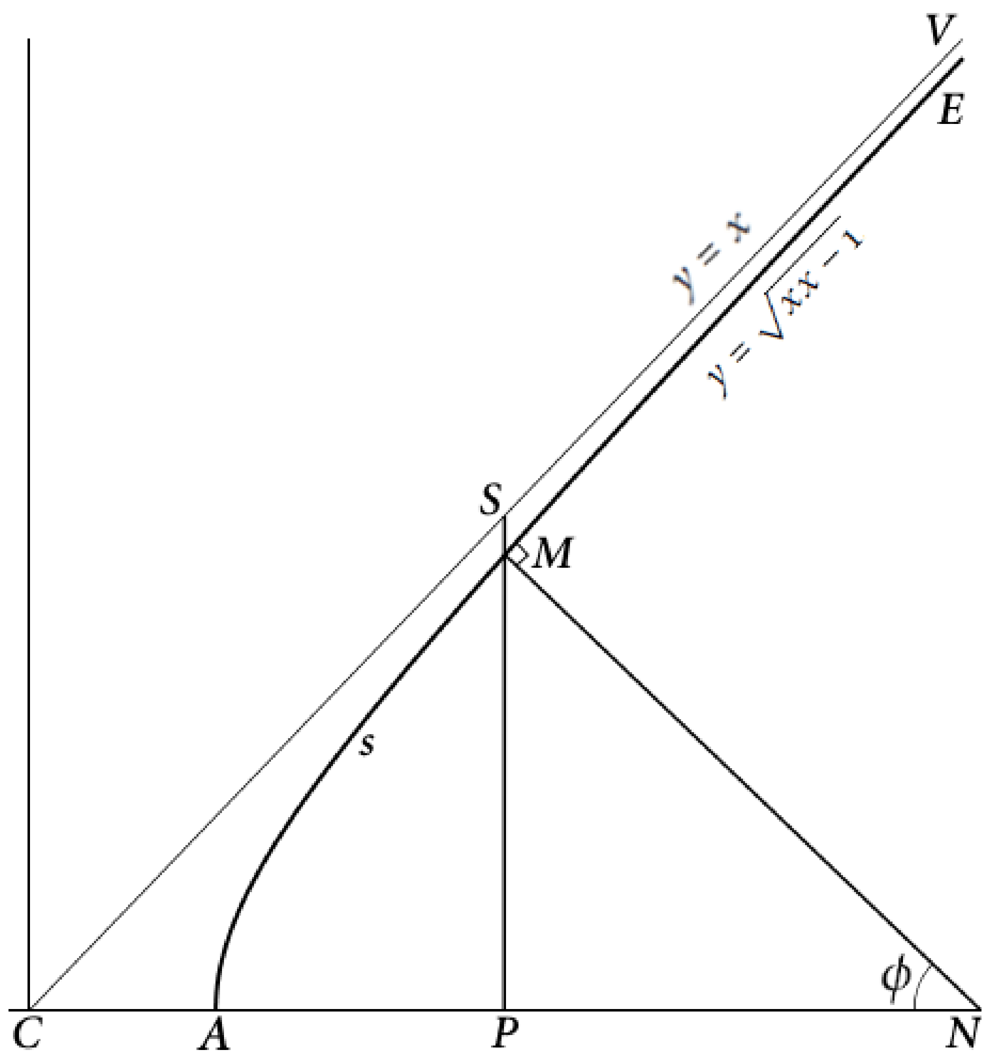

Figure 1[1]

§. 3. I can now easily express the length of the arc s by its amplitude ϕ. I set

$$s = \frac{z}{\sqrt{\cos 2\phi}}\,, \text{ and so } \partial s = \frac{\partial z \cos 2\phi + z\partial\phi \sin 2\phi}{\cos 2\phi\sqrt{\cos 2\phi}}\,,$$

whence the quantity z can be isolated from this equation:

$$\partial\phi = \partial z \cos 2\phi + z\partial\phi \sin 2\phi$$

through the integration of which I produce this series:

$$z = A \sin 2\phi + B \sin 6\phi + C \sin 10\phi + \text{etc.}$$

Indeed, it will soon be made clear that the angle must first be increased by 4ϕ. Since it must be that

$$1 = \frac{\partial z}{\partial\phi} \cos 2\phi + z \sin 2\phi\,,$$

we must first have that

$$\frac{\partial z}{\partial\phi} = 2A \cos 2\phi + 6B \cos 6\phi + 10C \cos 10\phi + \text{etc.}$$

from which, together with

$$\cos 2\phi \cos n\phi = \frac{1}{2} \cos(n-2)\,\phi + \frac{1}{2} \cos(n+2)\,\phi\,,$$

we see that

$$\begin{aligned}\tfrac{\partial z}{\partial\phi} \cos 2\phi = A &+ A \cos 4\phi + 3B \cos 8\phi + 5C \cos 12\phi \quad \text{etc.}\\ &+ 3B \qquad\quad + 5C \qquad\quad + 7D\end{aligned}$$

In the same way by putting

$$\sin 2\phi \sin n\phi = \frac{1}{2} \cos(n-2)\phi - \frac{1}{2} \cos(n+2)\phi$$

we recover

$$\begin{aligned}z \sin 2\phi = \tfrac{1}{2}A &- \tfrac{1}{2}A \cos 4\phi - \tfrac{1}{2}B \cos 8\phi - \tfrac{1}{2}C \cos 12\phi \quad \text{etc.}\\ &+ \tfrac{1}{2}B \qquad\quad + \tfrac{1}{2}C \qquad\quad + \tfrac{1}{2}D\end{aligned}$$

and therefore adding these series will produce another series with the equation:

$$\begin{aligned}1 = \tfrac{3}{2}A &+ \left(\tfrac{1}{2}A + \tfrac{7}{2}B\right) \cos 4\phi + \left(\tfrac{5}{2}B + \tfrac{11}{2}C\right) \cos 8\phi\\ &+ \left(\tfrac{9}{2}C + \tfrac{15}{2}D\right) \cos 12\phi + \left(\tfrac{13}{2}D + \tfrac{19}{2}E\right) \cos 16\phi + \text{etc.}\end{aligned}$$

§. 4. Thus the ordered sequence of coefficients is duly given by equality:

$$A = \frac{2}{3}, B = -\frac{1}{7}A, C = -\frac{5}{11}B, D = -\frac{9}{15}C, E = -\frac{13}{19}D, \text{etc.}$$

and by the substitution of these values we stumble upon this equation:

$$z = A\left(\sin 2\phi - \frac{1}{7}\sin 6\phi + \frac{1}{7}\cdot\frac{5}{11}\sin 10\phi - \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15}\sin 14\phi + \text{etc.}\right)$$

where I have factored $A = \frac{2}{3}$. This series is rendered $z = 0$ by the vanishing of ϕ, so that $s = 0$ by its very nature. On the other hand if this curve is extended to infinity, so that the curve might be confused with its asymptote, the amplitude will be $\phi = 45^\circ$, from which $\sin 2\phi = 1, \sin 6\phi = -1, \sin 10\phi = 1, \sin 14\phi = -1$, and so on. This being the case, the equation for z becomes:

$$z = A\left(1 + \frac{1}{7} + \frac{1}{7}\cdot\frac{5}{11} + \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15} + \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15}\cdot\frac{13}{19} + \text{etc.}\right)$$

the sum of which is clearly finite, yet the denominator goes to zero so that the arc s itself will have infinite magnitude, which is made evident from the equation $s = \frac{z}{\sqrt{\cos 2\phi}}$ since $\cos 2\phi = \cos 90^\circ = 0$.

§. 5. Refer now in the figure to the point E at the infinite end of the hyperbola, which corresponds asymptotically to the point V. The entire arc is given by $AE = \frac{z}{\sqrt{\cos 2\phi}}$, extending the amplitude so that $2\phi = 90^\circ$, and thus the interval $CS = x\sqrt{2} = \frac{\cos\phi\sqrt{2}}{\sqrt{\cos 2\phi}}$ is asymptotic as $\phi = 45^\circ$, and $CV = \frac{1}{\sqrt{\cos 2\phi}}$ will be infinite in length. On the other hand it is well known that the difference between the curve AME and the line CV is finite, and the curve AE is obviously less than the line CV, and (if a perpendicular AD is drawn from A to CV) greater than the line VD. Therefore let $CV - AE = \Delta$, where it suffices to know that Δ is a finite quantity. Therefore this gives $\frac{1-z}{\sqrt{\cos 2\phi}} = \Delta$, and so $z = 1 - \Delta\sqrt{\cos 2\phi}$ and consequently, since $\sqrt{\cos 2\phi} = 0$, we have $z = 1$. From this we may conclude, having discovered the sum of the series for z in the case that $\phi = 45^\circ$, that z is precisely 1. So now multiplying through by $\frac{3}{2}$ we have

$$1 + \frac{1}{7} + \frac{1}{7}\cdot\frac{5}{11} + \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15} + \text{etc.} = \frac{3}{2},$$

and thus subtracting 1 from both sides gives

$$\frac{1}{7} + \frac{1}{7}\cdot\frac{5}{11} + \frac{1}{7}\cdot\frac{5}{11}\cdot\frac{9}{15} + \text{etc.} = \frac{1}{2},$$

a series which I do not recall ever having seen committed to record.

§. 6. Therefore to verify the series arising from this summation, I carefully employed the following familiar method. I put

$$s = \frac{1}{7}x^7 + \frac{1}{7} \cdot \frac{5}{11}x^{11} + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15}x^{15} + \text{etc.}$$

so that putting $x = 1$ results in the above series. Therefore by differentiating we get:

$$\frac{\partial s}{\partial x} = x^6 + \frac{1}{7} \cdot 5x^{10} + \frac{1}{7} \cdot \frac{5}{11} \cdot x^{14} + \text{etc.}$$

Next it is true that

$$\frac{s}{xx} = \frac{1}{7}x^5 + \frac{1}{7} \cdot \frac{5}{11}x^9 + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15}x^{13} + \text{etc.}$$

and likewise by differentiating

$$\frac{1}{\partial x} \cdot \partial \cdot \frac{s}{xx} = \frac{5}{7}x^4 + \frac{1}{7} \cdot \frac{5}{11} \cdot 9x^8 + \frac{1}{7} \cdot \frac{5}{11} \cdot \frac{9}{15} \cdot 13x^{12} + \text{etc.}$$

which series, drawn to x^6 and subtracted from the former differentiated series yields $\frac{\partial s}{\partial x} - \frac{x^6}{\partial x} \cdot \partial \cdot \frac{x}{xx} = x^6$. It follows that the equation is finite, from which the value of s itself may be obtained.

§. 7. Following this fact produces the sum of this differential equation:

$$\partial s - x^4 \partial s + 2x^3 s \partial x = x^6 \partial x\,.$$

Dividing this equation by $s\left(1 - x^4\right)$ thus produces

$$\frac{\partial s}{s} + \frac{2x^3 \partial x}{1 - x^4} = \frac{x^6 \partial x}{s\left(1 - x^4\right)}\,,$$

which as an equation of the prior type is integrable. Indeed, the integral is

$$\log s - \frac{1}{2}\log 1 - x^4 = \log \frac{s}{\sqrt{1 - x^4}}\,,$$

which is obtained because:

$$\partial \log \frac{s}{\sqrt{1 - x^4}} = \frac{x^6 \partial x}{s\left(1 - x^4\right)}\,.$$

This integrable equation can then be multiplied by $\frac{s}{\sqrt{1-x^4}}$, whose right hand side may likewise be integrated. Thus we have

$$\frac{s}{\sqrt{1 - x^4}} \partial \log \frac{s}{\sqrt{1 - x^4}} = \frac{x^6 \partial x}{\left(1 - x^4\right)^{\frac{3}{2}}}\,,$$

which equation, by putting for brevity $\frac{s}{\sqrt{1-x^4}} = v$, is of this form:

$$d\partial \cdot \log v = \partial v = \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}},$$

whence therefore

$$v = \frac{s}{\sqrt{1-x^4}} = \int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}}.$$

§. 8. However it is easy to see that this last integral does not admit an algebraic solution. However in the mean time this equation reduces to a simpler form by employing the following method. Namely we put

$$\int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{\alpha x^3}{\sqrt{1-x^4}} + \beta \int \frac{xx\partial x}{\sqrt{1-x^4}},$$

and differentiating gives

$$\int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{3\alpha xx\partial x}{\sqrt{1-x^4}} + \frac{2\alpha x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} + \frac{\beta xx\partial x}{\sqrt{1-x^4}},$$

which equation multiplied through by

$$(1-x^4)^{\frac{3}{2}}$$

produces

$$x^6 = (3\alpha\beta)\, xx - (\alpha + \beta)\, x^6,$$

whence it is easily seen that we ought to set $\alpha + \beta = -1$ and $3\alpha + \beta = 0$, yielding $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{2}$, and thus we have

$$\int \frac{x^6 \partial x}{(1-x^4)^{\frac{3}{2}}} = \frac{x^3}{2\sqrt{1-x^4}} - \frac{3}{2} \int \frac{xx\partial x}{\sqrt{1-x^4}},$$

and so

$$s = \frac{1}{2}x^3 - \frac{3}{2}\sqrt{1-x^4} \int \frac{xx\partial x}{\sqrt{1-x^4}}.$$

§. 9. Nevertheless this last integral $\int \frac{xx\partial x}{\sqrt{1-x^4}}$ cannot be obtained. However, it is easy to see that for such an integral, for the case $x = 1$, the value must be finite, which suffices for our present situation. Therefore let this finite value be given by the formula $\int \frac{xx\partial x}{\sqrt{1-x^4}} = \Delta$, and by putting $x = 1$ the discovered equation will give $s = \frac{1}{2}$, which is the sum that our earlier series supplied.

§. 10. Therefore a similar operation may be employed to extend this method to many other summations of more general series, so we undetake the following problem:

Problem.

To find the sum of the following infinite series:

$$\frac{a}{b}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\cdot\frac{a+2\theta}{b+2\theta}+\text{etc.}$$

Solution.

§. 11. We state as before

$$s=\frac{a}{b}x^{b}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}x^{b+\theta}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\cdot\frac{a+2\theta}{b+2\theta}x^{b+2\theta}+\text{etc.}$$

in which, therefore, we examine the case $x=1$. Now truly when differentiated this series will give

$$\frac{\partial s}{\partial x}=ax^{b-1}+\frac{a}{b}\cdot(a+\theta)\,x^{b+\theta-1}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\cdot(a+2\theta)\,x^{b+2\theta-1}+\text{etc.}$$

Then multiply the series itself by $x^{a-b+\theta}$, giving

$$x^{a-b+\theta}\cdot s=\frac{a}{b}x^{a+\theta}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}x^{a+2\theta}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\cdot\frac{a+2\theta}{b+2\theta}x^{a+3\theta}+\text{etc.}$$

which by differentiating yields:

$$\frac{1}{\partial x}\partial\cdot x^{a-b+\theta}\cdot s=\frac{a}{b}\,(a+\theta)\,x^{a+\theta-1}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\,(a+2\theta)\,x^{a+2\theta-1}+\frac{a}{b}\cdot\frac{a+\theta}{b+\theta}\cdot\frac{a+2\theta}{b+2\theta}\,(a+3\theta)\,x^{a+3\theta-1}+\text{etc.}$$

Multiplying by x^{b-a} and subtracting from the above eliminates all of the terms of this series after the first; giving

$$\frac{\partial s}{\partial x}-\frac{x^{b-a}}{\partial x}\partial\cdot x^{a-b+\theta}\cdot s=ax^{b-1},$$

which is, therefore, a finite equation, from which the unknown s may be obtained.

§. 12. Therefore the fact may be reduced to this differential equation:

$$\partial s\,(1-x)-(a-b+\theta)\,sx^{\theta-1}\partial x=ax^{b-1}\partial x,$$

which we divide by $s\left(1-x^{\theta}\right)$, so that we might obtain

$$\frac{\partial s}{s}-\frac{(a-b+\theta)\,x^{\theta-1}\partial x}{\left(1-x^{\theta}\right)}=\frac{ax^{b-1}\partial x}{s\left(1-x^{\theta}\right)},$$

where the integral of the left term is

$$\log s+\frac{a-b+\theta}{\theta}\log\left(1-x^{\theta}\right),\ \text{or}$$

$$\log s - \frac{b-a-\theta}{\theta}\log\left(1-x^{\theta}\right).$$

We will soon see that the sum of this series will not be finite, unless $b > a+\theta$. To this end we put

$$\frac{s}{\left(1-x^{\theta}\right)^{\frac{b-a-\theta}{\theta}}} = v$$

so that the left member gives $\partial \log v$, and the equation becomes

$$\partial \cdot \log v = \frac{ax^{b-1}\partial x}{s\left(1-x^{\theta}\right)},$$

which multiplied by v avoids the integration, giving

$$v \cdot \partial \log v = \partial v = \frac{ax^{b-1}\partial x}{\left(1-x^{\theta}\right)^{\frac{b-a}{\theta}}},$$

of which the integral is

$$v = \frac{s}{\left(1-x^{\theta}\right)^{\frac{b-a-\theta}{\theta}}} = a\int \frac{x^{b-1}\partial x}{\left(1-x^{\theta}\right)^{\frac{b-a}{\theta}}}.$$

§. 13. For the sake of brevity we put $\frac{b-a-\theta}{\theta} = n$, so that $b = n\theta + a + \theta$, or $a = b - \theta - n\theta$, and thus our equation becomes

$$\frac{s}{\left(1-x^{\theta}\right)^{n}} = a\int \frac{x^{b-1}\partial x}{\left(1-x^{\theta}\right)^{n+1}}.$$

Now we may employ the reasoning seized upon above, putting

$$\int \frac{x^{b-1}\partial x}{\left(1-x^{\theta}\right)^{n+1}} = \frac{\alpha x^{b-\theta}}{\left(1-x^{\theta}\right)^{n}} + \beta\int \frac{x^{b-\theta-1}\partial x}{\left(1-x^{\theta}\right)^{n}},$$

which by differentiating gives

$$\frac{x^{b-1}}{\left(1-x^{\theta}\right)^{n+1}} = \frac{\alpha\,(b-\theta)\,x^{b-\theta-1}}{\left(1-x^{\theta}\right)^{n}} + \frac{n\alpha\theta x^{b-1}}{\left(1-x^{\theta}\right)^{n+1}} + \frac{\beta x^{b-\theta-1}}{\left(1-x^{\theta}\right)^{n}},$$

which equation, multiplied by $\left(1-x^{\theta}\right)^{n+1}$, gives:

$$x^{b-1} = \left[\alpha\,(b-\theta) + \beta\right]x^{b-\theta-1} - \left[\alpha\,(b-\theta) - n\alpha\theta + \beta\right]x^{b-1},$$

from which it follows that we ought to let $\alpha = \frac{1}{n\theta}$ and $\beta = -\frac{b-\theta}{n\theta}$. Thus this equation for x, multiplied by $\left(1-x^{\theta}\right)^{n+1}$, will give

$$s = \frac{a}{n\theta}x^{b-\theta} - \frac{a\,(b-\theta)}{n\theta}\left(1-x^{\theta}\right)^{n}\int \frac{x^{b-\theta-1}\partial x}{\left(1-x^{\theta}\right)^{n}}.$$

§. 14. On the other hand because this is true for the right hand side, even though the summation cannot be determined, because it has $\left(1 - x^{\theta}\right)^{n}$ in the denominator of the differential it is certain that in the integral, as one can show, the denominator will contain $\left(1 - x^{\theta}\right)^{n-1}$, which obviously has power one less than the preceding. So one is safe to assume that the integral will have the form $\frac{Q}{\left(1-x^{\theta}\right)^{n-1}}$, where it suffices to know that the denominator in Q does not contain any more than $1 - x^{\theta}$. By the substitution of this value we have:

$$s = \frac{a}{n\theta} x^{b-\theta} - \frac{a\,(b-\theta)}{n\theta} \left(1 - x^{\theta}\right) Q\,.$$

§. 15. Now having examined the sum of the series more generally, we will coax the sum of the proposed series, by letting $x = 1$. Moreover it will be shown that $s = \frac{a}{n\theta}$, so that the unknown quantity Q utterly disappears. Therefore for the sake of brevity we let $n = \frac{b-a-\theta}{\theta}$, so that the sum of our series $= \frac{\alpha}{b-a-\theta}$, from which we deduce the following

Theorem.

If this infinite series be proposed:

$$\frac{a}{b} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} + \frac{a}{b} \cdot \frac{a+\theta}{b+\theta} \cdot \frac{a+2\theta}{b+2\theta} + \text{etc.}$$

its sum is always $= \frac{a}{b-a-\theta}$*; from which it follows, unless it be that* $b > a + \theta$*, that this series does not have a finite sum, but will be infinite.*

§. 16. Therefore, with this sum having been broadly stated, it is helpful to observe that it contains a most well-known series, namely that which is born from the expansion of the binomial. If we expand the binomial $(1 - x)^{\frac{a}{b}}$, and let $b > a$, it will form the following series:

$$(1 - x)^{\frac{a}{b}} = 1 - \frac{a}{b} x + \frac{a\,(b-a)}{b \cdot 2b} x^2 - \frac{a\,(b-a)\,(2b-a)}{b \cdot 2b \cdot 3b} x^3 + \text{etc.}$$

Thus if we let $x = 1$, we obtain this series

$$1 = \frac{a}{b} + \frac{a}{b} \cdot \frac{b-a}{2b} + \frac{a}{b} \cdot \frac{b-a}{2b} \cdot \frac{2b-a}{3b} + \text{etc.}$$

which agrees remarkably well with our theorem. If we multiply it by $\frac{b}{a}$, it will give

$$\frac{b}{a} = 1 + \frac{b-a}{2b} + \frac{b-a}{2b} \cdot \frac{2b-a}{3b} + \text{etc.}\,,$$

and therefore

$$\frac{b}{a}-1=\frac{b-a}{2b}+\frac{b-a}{2b}\cdot\frac{2b-a}{3b}+\text{etc.}$$

If we compare this series with our general one, we see that ours had a where this has $b-a$, where ours had b this has ab, and where ours had θ this has b. Further when our sum had $\frac{a}{b-a-\theta}$, the present sum has now $\frac{b-a}{a}$, which agrees most beautifully.

§. 17. Moreover this can be applied to the sum of more general series, in which an infinite number of variables occur, as I will show in the following theorem.

General Theorem.

If the variables a, b, c, d, etc. with θ denote as many values as are needed by this series, whether it runs on infinitely, or terminates at any point:

$$\frac{a}{b+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}\cdot\frac{c}{d+\theta}+\text{etc.}$$

the sum is always $\frac{a}{\theta}$.

Truly that method shown above is in no way permitted by Analysis; however from the principles of common Algebra will I exhibit two proofs: the former from the composition of the series itself, whence I will derive the expression $\frac{a}{\theta}$ as the sum of the proposed series, and the other from a consideration of the sum of the series.

First Proof.

§18. This proof is most obviously obtained from the following considerations. Namely put:

$$\begin{aligned}
&\text{I}^\circ.\ \tfrac{a}{\theta}=\tfrac{a}{b+\theta}+\tfrac{p}{\theta}\ \text{, which gives }\ p=\tfrac{ab}{b+\theta}\ ,\\
&\text{II}^\circ.\ \tfrac{p}{\theta}=\tfrac{p}{c+\theta}+\tfrac{q}{\theta}\ \text{, which gives }\ q=\tfrac{cp}{c+\theta}\ ,\\
&\text{III}^\circ.\ \tfrac{q}{\theta}=\tfrac{q}{d+\theta}+\tfrac{r}{\theta}\ \text{, which gives }\ r=\tfrac{dq}{d+\theta}\ ,\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

proceeding in this way as long as needed.

§. 19. Now if we here substitute for the constant p, we will have this equation: $\frac{a}{\theta}=\frac{a}{b+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{\theta}$. Further, if we substitute for the constant q, it gives $\frac{p}{\theta}=\frac{p}{c+\theta}+\frac{cp}{(c+\theta)\theta}$, so that

$$\frac{a}{\theta}=\frac{a}{b+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}\cdot\frac{c}{\theta}.$$

If still further we substitute for the constant r it gives:

$$\frac{a}{\theta}=\frac{a}{b+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}\cdot\frac{c}{d+\theta}+\frac{a}{b+\theta}\cdot\frac{b}{c+\theta}\cdot\frac{c}{d+\theta}\cdot\frac{d}{\theta}$$

and so on. From this it is clear that the sum, continued even to an infinite number of terms, will always have a value. Indeed this is so, since the final term differs from the residue by such a small amount that this infinitessimal term clearly should not be counted. For, since all the factors in the numerator are smaller than those in the denominator, it is obvious that the value of the infinitessimal term all but disappears.

Alternate Proof.

§. 20. Now I seek to examine the sum of this series, which $= s$. To this end I split the final factors of each term into two parts, in the following manner:

$$\begin{aligned}
\text{I}^\circ.\ \tfrac{a}{b+\theta} &= \tfrac{a}{\theta}-\tfrac{ab}{(b+\theta)\theta}\,,\\
\text{II}^\circ.\ \tfrac{b}{c+\theta} &= \tfrac{b}{\theta}-\tfrac{bc}{(c+\theta)\theta}\,,\\
\text{III}^\circ.\ \tfrac{c}{d+\theta} &= \tfrac{c}{\theta}-\tfrac{cd}{(d+\theta)\theta}\,,\\
&\text{etc.}
\end{aligned}$$

§. 21[2]. Now if I introduce values in place of the final terms of our series by dividing them into two factors, I obtain the following:

$$\begin{aligned}
s = {}& \tfrac{a}{\theta} + \tfrac{a}{b+\theta}\cdot\tfrac{b}{\theta} + \tfrac{a}{b+\theta}\cdot\tfrac{b}{c+\theta}\cdot\tfrac{c}{\theta} \\
& - \tfrac{ab}{(b+\theta)\theta} - \tfrac{a}{b+\theta}\cdot\tfrac{bc}{(c+\theta)\theta} - \tfrac{a}{b+\theta}\cdot\tfrac{b}{c+\theta}\cdot\tfrac{cd}{(d+\theta)\theta} \quad \text{etc.}
\end{aligned}$$

where clearly each negative term is cancelled by a positive, so that the first positive and final negative term are all that remain. However if this series is carried through to infinity, in the manner I have observed, the final term will go to zero, the numerator being infinitely less than the denominator. Therefore the entire sum can be rewritten as $s=\frac{a}{\theta}$.

§. 22. This final sum I recall having already posed to my rivals: however I cannot remember it ever having been released to the public. Indeed in an exchange of letters forty years ago with the late honorable Goldbach III, these matters frequently arose, wherefore I do not doubt but that geometers will find this discovery benign.

[2] *Translator's note:* In the original manuscript, this section is mistakenly numbered as 20.