INNUMERAE AEQUATIONUM FORMAE EX OMNIBUS ORDINIBUS QUARUM RESOLUTIO EXHIBERI POTEST

INFINITELY MANY FORMS OF EQUATIONS OF ALL ORDERS, THE SOLUTION OF WHICH CAN BE EXHIBITED

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Presented to the assembly on 6 May, 1776.

Nova acta academiae scientiarum Petropolitanae **6** (1788), 1790, pp. 25-35 [E644]. The Summary (in French) appears on pp. 82-84 of the same issue.[1]

Summary

The general rules given for the solution of equations not being applicable for equations that go beyond the fourth degree, it is of the utmost importance to distinguish those equations of higher orders that can be solved. Their number is infinitely large. Several Geometers, and particularly the Author himself of this Treatise, have previously cited some solvable equations of all degrees, to say nothing of either equations that have rational roots or of those which may be resolved into factors and reduced thus to the forms of equations solvable by the general rules.

In the present Treatise, the late M. EULER has given again an infinite number of algebraic equations of which all the roots can be assigned. All these equations are contained in the following general form

$$x^n = n''ab\left(\frac{a-b}{a-b}\right)x^{n-2} + n'''ab\left(\frac{a^2-b^2}{a-b}\right)x^{n-3} + n''''ab\left(\frac{a^3-b^3}{a-b}\right)x^{n-4} + \text{etc.},$$

where the letters n'', n''', n'''' etc. indicate the second, third, fourth etc. coefficient of the binomial raised to the n^{th} power [2]; and one of the roots of this equation of the order n is

$$\frac{b\sqrt[n]{\frac{a}{b}} - a}{1 - \sqrt[n]{\frac{a}{b}}},$$

[1]The translators have incorporated the footnotes by Paul Stäckel in the version that appears in Vol. I.6 of Euler's *Opera Omnia*.

[2]Translators: That is, $n'' = \frac{n(n-1)}{1\cdot 2}$, which is the coefficient of x^{n-2} in the expansion of $(1+x)^n$; and $n''' = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$, which is the coefficient of x^{n-3} in that expansion; and so on.

where $\sqrt[n]{\frac{a}{b}}$ admits n different values so that one will also obtain all the n roots of the mentioned equation of the n^{th} degree.

This truth is drawn from the consideration of this equation $(\frac{a+x}{b+x})^n = \frac{a}{b}$; for one extracts thence $\frac{a+x}{b+x} = \sqrt[n]{\frac{a}{b}}$ and proceeding

$$x = \frac{b\sqrt[n]{\frac{a}{b}} - a}{1 - \sqrt[n]{\frac{a}{b}}},$$

and developing thence the assumed equation

$$b(a+x)^n = a(b+x)^n$$

it will take the form expounded above

$$x^n = n''ab\left(\frac{a-b}{a-b}\right)x^{n-2} + n'''ab\left(\frac{a^2-b^2}{a-b}\right)x^{n-3} + \text{etc.}$$

One could conjecture that the consideration of this equation $\left(\frac{f+x}{g+x}\right)^n = \frac{a}{b}$ could lead to some different and more general forms of resolvable equations; but the Author, to prevent this error, shows that however different from a and b may be the quantities f and g, this equation always allows itself to be reduced to the previous [one].

Finally, M. EULER observes in the forms of roots found for the equations contained in his general equation, a new confirmation of the conjecture that he had formerly proposed,[3] concerning the solution of equations in which the second term is absent, as for example, in this

$$x^n = px^{n-2} + qx^{n-3} + rx^{n-4} + \text{etc.},$$

while maintaining that any such equations were always solvable by means of a resolvent equation[4] of one degree lower of the form

$$y^{n-1} - Ay^{n-2} + By^{n-3} - Cy^{n-4} + \text{etc.} = 0,$$

the form of the root being

$$x = \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \text{etc.},$$

where α, β, γ, δ etc. indicate the roots of the resolvent equation, which are $n-1$ in number.

[3]Stäckel points to E30.

[4]Translators: In translating the phrase as "a resolvent equation", we have corrected a misprint in the version of the summary that appears in Euler's *Opera Omnia*, which reads "une équations résolvante", whereas the summary in *Nova acta academiae scientiarum Petropolitanae* reads "une équation résolvante".

Infinitely Many Forms of Equations of All Orders, the Solution of Which Can Be Exhibited

1. Since general rules for the solution of equations do not extend beyond the fourth degree, it will be of the greatest import to have noted, of equations of this type, the forms that permit solutions. Here, however, I speak of equations of this type, which neither have rational roots nor can be solved through factors into equations of lower degrees, since it would be very easy to produce infinitely many solvable equations of this kind. For this reason, the appearances of equations of this kind ought to be thought worthy of attention, whose solution necessarily demands the extraction of roots of the same order as the equation itself.

2. Equations of this type have previously been published by de Moivre[5] for individual orders, by means of which analytical knowledge should deservedly be thought to have been amplified in no small measure; thereupon indeed I myself have brought to light more such equations;[6] recently, however, a method has occurred to me of eliciting countless other equations of this kind, which I hope will be not at all unwelcome to Geometers.

3. Therefore, I shall propose these forms of equations, just as I have been led to them, here in order.

I. If $x^2 = ab$, it will be the case that $x = \sqrt{ab}$.

II. If $x^3 = 3abx + ab(a+b)$, it will be the case that $x = \sqrt[3]{aab} + \sqrt[3]{abb}$.

III. If $x^4 = 6abxx + 4ab(a+b)x + ab(aa+ab+bb)$, it will be the case that

$$x = \sqrt[4]{a^3b} + \sqrt[4]{aabb} + \sqrt[4]{ab^3}.$$

IV. If

$$x^5 = 10abx^3 + 10ab(a+b)xx + 5ab(aa+ab+bb)x + ab(a^3+aab+abb+b^3),$$

it will be the case that

$$x = \sqrt[5]{a^4b} + \sqrt[5]{a^3bb} + \sqrt[5]{aab^3} + \sqrt[5]{ab^4}.$$

V. If

$$\begin{aligned} x^6 &= 15abx^4 + 20ab(a+b)x^3 + 15ab(aa+ab+bb)xx \\ &\quad + 6ab(a^3+aab+abb+b^3)x + ab(a^4+a^3b+aabb+ab^3+b^4), \end{aligned}$$

it will be the case that

$$x = \sqrt[6]{a^5b} + \sqrt[6]{a^4bb} + \sqrt[6]{a^3b^3} + \sqrt[6]{aab^4} + \sqrt[6]{ab^5}.$$

VI. If

$$\begin{aligned} x^7 &= 21abx^5 + 35ab(a+b)x^4 + 35ab(aa+ab+bb)x^3 + 21ab(a^3+aab+abb+b^3)xx \\ &\quad + 7ab(a^4+a^3b+aabb+ab^3+b^4)x + ab(a^5+a^4b+a^3bb+aab^3+ab^4+b^5), \end{aligned}$$

[5] Stäckel points to A. de Moivre's *Aequationum Quarundam Potestatis Tertiae, Quintae, Septimae, Nonae, et Superiorum, ad Infinitum Usque Pergendo, in Terminis Finitis, ad Instar Regularum pro Cubicis Quae Vocantur Cardani, Resolutio Analytica*, Philosophical Transactions (London) **25**, 1707, pp. 2368-2371.

[6] Stäckel points to §41 and the subsequent sections of E282.

it will be the case that

$$x = \sqrt[7]{a^6b} + \sqrt[7]{a^5bb} + \sqrt[7]{a^4b^3} + \sqrt[7]{a^3b^4} + \sqrt[7]{aab^5} + \sqrt[7]{ab^6}.$$

4. Hence it is now easily inferred that in general for any order

$$\begin{aligned} x^n = {} & \frac{n(n-1)}{1\cdot 2}abx^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}ab(a+b)x^{n-3} \\ & + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}ab(aa+ab+bb)x^{n-4} \\ & + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}ab(a^3+aab+abb+b^3)x^{n-5} + \text{etc.} \end{aligned}$$

it will be the case that

$$x = \sqrt[n]{a^{n-1}b} + \sqrt[n]{a^{n-2}bb} + \sqrt[n]{a^{n-3}b^3} + \sqrt[n]{a^{n-4}b^4} + \text{etc.}$$

Or if we should write in place of those coefficients, for the sake of brevity, n^{II}, n^{III}, n^{IV}, n^{V}, n^{VI} etc., this general equation will be able to be more succinctly expressed as follows

$$\begin{aligned} x^n = {} & n^{II}ab\left(\frac{a-b}{a-b}\right)x^{n-2} + n^{III}ab\left(\frac{a^2-b^2}{a-b}\right)x^{n-3} + n^{IV}ab\left(\frac{a^3-b^3}{a-b}\right)x^{n-4} \\ & + n^{V}ab\left(\frac{a^4-b^4}{a-b}\right)x^{n-5} + n^{VI}ab\left(\frac{a^5-b^5}{a-b}\right)x^{n-6} + \text{etc.}; \end{aligned}$$

then indeed the root itself will also be able to be more elegantly expressed, so that it is the case that

$$x = \frac{a\sqrt[n]{b} - b\sqrt[n]{a}}{\sqrt[n]{a} - \sqrt[n]{b}},$$

which therefore is a general equation extending to all orders.

5. One may change these equations into another form whereby the artifice which has led to that result is more concealed. Indeed let us set the product of the letters a and b, $ab = p$ and their sum $a + b = s$, and introduce these two letters p and s in place of a and b into the computation; then, moreover, it will be the case that

$$a = \frac{s + \sqrt{ss - 4p}}{2} \quad \text{and} \quad b = \frac{s - \sqrt{ss - 4p}}{2}.$$

With the introduction now of these new values the special equations above will take on the following forms:

I. If $x^2 = p$, it will be the case that $x = \sqrt{p}$.

II. If $x^3 = 3px + ps$, it will be the case that

$$x = \sqrt[3]{aab} + \sqrt[3]{abb} = \sqrt[3]{ap} + \sqrt[3]{bp}.$$

III. If $x^4 = 6pxx + 4psx + p(ss - p)$, it will be the case that

$$x = \sqrt[4]{aap} + \sqrt[4]{abp} + \sqrt[4]{bbp}.$$

IV. If $x^5 = 10px^3 + 10psxx + 5p(ss - p)x + p(s^3 - 2sp)$, it will be the case that

$$x = \sqrt[5]{a^3p} + \sqrt[5]{ap^2} + \sqrt[5]{bp^2} + \sqrt[5]{b^3p}.$$

V. If $x^6 = 15px^4 + 20psx^3 + 15p(ss - p)xx + 6p(s^3 - 2ps)x + p(s^4 - 3ps^2 + pp)$, it will be the case that

$$x = \sqrt[6]{a^4p} + \sqrt[6]{aapp} + \sqrt[6]{p^3} + \sqrt[6]{bbpp} + \sqrt[6]{b^4p}.$$

VI. If

$$\begin{aligned} x^7 &= 21px^5 + 35psx^4 + 35p(ss - p)x^3 + 21p(s^3 - 2ps)xx \\ &\quad + 7p(s^4 - 3pss + pp)x + p(s^5 - 4ps^3 + 3pps), \end{aligned}$$

it will be the case that

$$x = \sqrt[7]{a^5p} + \sqrt[7]{a^3pp} + \sqrt[7]{ap^3} + \sqrt[7]{bp^3} + \sqrt[7]{b^3pp} + \sqrt[7]{b^5p}$$

etc.

6. So that we may now restore this general form, we must observe that the new coefficients contained in the letters p and s constitute a recurrent series whose scale of relation[7] is s, $-p$. For if we set

$$Q = \frac{a^\lambda - b^\lambda}{a - b}, \qquad Q' = \frac{a^{\lambda+1} - b^{\lambda+1}}{a - b} \qquad \text{and} \qquad Q'' = \frac{a^{\lambda+2} - b^{\lambda+2}}{a - b},$$

it will clearly be the case that

$$Q'' = sQ' - pQ;$$

for because $s = a + b$, it will be the case that

$$sQ' = \frac{a^{\lambda+2} + ba^{\lambda+1} - ab^{\lambda+1} - b^{\lambda+2}}{a - b},$$

but because $p = ab$, it will be the case that

$$pQ = \frac{a^{\lambda+1}b - ab^{\lambda+1}}{a - b},$$

which form having been subtracted from the previous one, what remains will be

[7]Translators: That is, if Q, Q', and Q'' are consecutive terms in the sequence $\{\frac{a^\lambda - b^\lambda}{a-b}\}$, then Euler claims that Q'' can be obtained from the two preceding terms by means of the equation $Q'' = sQ' - pQ$. Regarding recurrent series and indices or scales of relations, Stäckel refers the reader to two works by A. de Moivre (1667-1754): *De fractionibus algebraicis radicalitate immunibus ad fractiones simpliciores reducendis, deque summandis terminis quarumdam serierum aequali intervallo a se distantibus*, Philosophical Transactions (London) **32** (1722/3), 1724, numb. 373, p. 162, especially p. 176; and *Miscellanea analytica de seriebus et quadraturis*, London, 1730, p. 27. Stäckel also cites Vol. 1 of Euler's *Introductio in analysin infinitorum* [E101], Chapters IV, XIII, and XVII.

$$sQ' - pQ = \frac{a^{\lambda+2} - b^{\lambda+2}}{a-b}.$$

Thus, having observed this law, we shall have the following transformations:

$$\begin{array}{r|l}
\frac{a-b}{a-b}=1, & \frac{a^5-b^5}{a-b}=s^4-3pss+pp, \\
\frac{a^2-b^2}{a-b}=s, & \frac{a^6-b^6}{a-b}=s^5-4ps^3+3pps, \\
\frac{a^3-b^3}{a-b}=ss-p, & \frac{a^7-b^7}{a-b}=s^6-5ps^4+6ppss-p^3, \\
\frac{a^4-b^4}{a-b}=s^3-2sp, & \frac{a^8-b^8}{a-b}=s^7-6ps^5+10pps^3-4p^3s
\end{array}$$

etc.

7. The order by which these formulae progress is now sufficiently clear. For first the powers of this s decrease constantly by two, whereas the powers of this p increase by one with alternating signs;[8] moreover, the numeric coefficients of each term coincide with those which the same terms would have[9] in the development of the binomial, or what amounts to the same thing, they indicate all permutations of the letters p and s, so that the coefficient of the term $p^\alpha s^\beta$ is

$$= \frac{1 \cdot 2 \cdot 3 \cdots (\alpha+\beta)}{1 \cdot 2 \cdot 3 \cdots \alpha \cdot 1 \cdot 2 \cdot 3 \cdots \beta}.$$

Hence then we have deduced the following general transformation

$$\begin{aligned}
\frac{a^{\lambda+1} - b^{\lambda+1}}{a-b} = {} & s^\lambda - \frac{\lambda-1}{1} ps^{\lambda-2} + \frac{(\lambda-2)(\lambda-3)}{1 \cdot 2} pps^{\lambda-4} \\
& - \frac{(\lambda-3)(\lambda-4)(\lambda-5)}{1 \cdot 2 \cdot 3} p^3 s^{\lambda-6} + \frac{(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{1 \cdot 2 \cdot 3 \cdot 4} p^4 s^{\lambda-8} \\
& - \frac{(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} p^5 s^{\lambda-10} + \text{etc.}
\end{aligned}$$

8. But if then we should substitute these values in the general equation given in §4 above,[10] the general equation whose solution one may display by this method will have such a form as this

[8]Translators: That is, as one moves from one term to the next in the expression for $\frac{a^\lambda - b^\lambda}{a-b}$, the power of s goes down by 2 and the power of p increases by 1, with the signs of the terms alternating.

[9]Translators: We translate "essent habiturae" as if the Latin were "essent habituri," to agree with "iidem termini".

[10]Translators: That is, substitute the values for $\frac{a^\lambda - b^\lambda}{a-b}$ into the second form of the general equation in §4.

$$x^n = \frac{n(n-1)}{1\cdot 2}px^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}psx^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}p(ss-p)x^{n-4}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}p(s^3-2ps)x^{n-5}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}p(s^4-3pss+pp)x^{n-6}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}p(s^5-4ps^3+3pps)x^{n-7} + \text{etc.}$$

Clearly the solution of this equation, whatever numbers may be taken for p and s, will always be possible, for indeed its root, after these have been derived from the numbers p and s

$$a = \frac{s+\sqrt{ss-4p}}{2} \quad \text{and} \quad b = \frac{s-\sqrt{ss-4p}}{2},$$

will be expressed so that

$$x = \frac{a\sqrt[n]{b} - b\sqrt[n]{a}}{\sqrt[n]{a} - \sqrt[n]{b}}.$$

9. Indeed this formula gives us a single root of the proposed equation, but yet all roots of the same equation clearly are easily deduced from it, the number of which is $= n$. For first, letting $b = ak$, that root will be restored to a single radical sign, since hereby it becomes

$$x = \frac{a\sqrt[n]{k} - b}{1 - \sqrt[n]{k}}.$$

Now in fact that root, namely $\sqrt[n]{k}$, admits distinct values numbering n, just as also the root of the power n of unity, namely $\sqrt[n]{1}$, receives as many different values, of which one value is always equal to unity itself. Whence if any of these values be designated by the letter ρ, so that $\rho^n = 1$, that letter ρ will include n different values, any one of which one may join with the formula $\sqrt[n]{k}$, namely by writing in its place $\rho\sqrt[n]{k}$, wherefore plainly all the roots of the proposed equation will be contained in this formula

$$x = \frac{\rho a\sqrt[n]{k} - b}{1 - \rho\sqrt[n]{k}} \quad \text{or} \quad x = \frac{\rho a\sqrt[n]{b} - b\sqrt[n]{a}}{\sqrt[n]{a} - \rho\sqrt[n]{b}};$$

then truly if this formula be developed by division, the following expression will result

$$x = \rho\sqrt[n]{a^{n-1}b} + \rho^2\sqrt[n]{a^{n-2}bb} + \rho^3\sqrt[n]{a^{n-3}b^3} + \text{etc.},$$

of which expression the number of terms is $n-1$, with the last being $\rho^{n-1}\sqrt[n]{ab^{n-1}}$.

10. It will be worth the effort to have illustrated this matter with an example. Let us then suppose $n = 5$, $s = 1$ and $p = -1$, so that this equation of the fifth degree is proposed

$$x^5 = -10x^3 - 10xx - 10x - 3$$

or

$$x^5 + 10x^3 + 10xx + 10x + 3 = 0.$$

Thus to discover the roots of this equation, let these values be taken

$$a = \frac{1+\sqrt{5}}{2} \quad\text{and}\quad b = \frac{1-\sqrt{5}}{2},$$

upon the discovery of which, any root will be

$$x = \frac{\rho a\sqrt[5]{b} - b\sqrt[5]{a}}{\sqrt[5]{a} - \rho\sqrt[5]{b}},$$

or, by introducing the letter $k = \frac{b}{a} = \frac{-3+\sqrt{5}}{2}$, the result will be

$$x = \frac{\rho a\sqrt[5]{k} - b}{1 - \rho\sqrt[5]{k}}.$$

But if however we should wish to develop this form, because $ab = p = -1$, we shall find that

$$x = -\rho\sqrt[5]{a^3} + \rho^2\sqrt[5]{a} + \rho^3\sqrt[5]{b} - \rho^4\sqrt[5]{b^3},$$

which expression, when completely developed in numbers, produces

$$x = -\rho\sqrt[5]{2+\sqrt{5}} + \rho^2\sqrt[5]{\frac{1+\sqrt{5}}{2}} + \rho^3\sqrt[5]{\frac{1-\sqrt{5}}{2}} - \rho^4\sqrt[5]{2-\sqrt{5}}.$$

A DEMONSTRATION OF THE FORMULAE GIVEN ABOVE

11. The analysis that has led to these equations is very much accessible, such that it seems to have scarcely anything mysterious about it; for it has been entirely derived from this very simple equation

$$\frac{(a+x)^n}{(b+x)^n} = \frac{a}{b}.$$

For since from this it turns out that

$$\frac{a+x}{b+x} = \sqrt[n]{\frac{a}{b}},$$

thence is deduced the unknown [11]

$$x = \frac{a - b\sqrt[n]{\frac{a}{b}}}{\sqrt[n]{\frac{a}{b}} - 1} = \frac{a\sqrt[n]{b} - b\sqrt[n]{a}}{\sqrt[n]{a} - \sqrt[n]{b}},$$

which is the very root that we have assigned for the equations above.

[11] Translators: The x on the left side of this equation appears in the paper as published in *Nova acta academiae scientiarum Petropolitanae* but was omitted in the version in Euler's *Opera Omnia*.

12. But if truly we should develop this assumed equation, since thence it ought to become $a(x+b)^n = b(x+a)^n$ or $a(x+b)^n - b(x+a)^n = 0$, from this the following equation will be derived

$$\left.\begin{aligned} ax^n + \frac{n}{1}abx^{n-1} + \frac{n(n-1)}{1\cdot 2}abbx^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}ab^3x^{n-3} + \text{etc.} \\ -bx^n - \frac{n}{1}abx^{n-1} - \frac{n(n-1)}{1\cdot 2}aabx^{n-2} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3bx^{n-3} - \text{etc.} \end{aligned}\right\} = 0,$$

where the second members are mutually removed.[12] Now because the first member is affected by $a-b$, the remaining members may be transferred into the other part and divided by $a-b$, and so the following equation will emerge[13]

$$\begin{aligned} x^n = \frac{n(n-1)}{1\cdot 2}ab\left(\frac{a-b}{a-b}\right)x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}ab\left(\frac{aa-bb}{a-b}\right)x^{n-3} \\ + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}ab\left(\frac{a^3-b^3}{a-b}\right)x^{n-4} + \text{etc.}, \end{aligned}$$

which is the very general equation treated above, whose root then is

$$x = \frac{a\sqrt[n]{b} - b\sqrt[n]{a}}{\sqrt[n]{a} - \sqrt[n]{b}}.$$

13. Hence perhaps someone might be able to expect that equations of this kind that are more general can be obtained in a similar way, if in place of that very simple formula, this more widely extending formula

$$\frac{(f+x)^n}{(g+x)^n} = \frac{a}{b}$$

should be established as a basis, if indeed here four arbitrary quantities a, b, f and g are introduced into the calculation, whereas previously only two a and b were present; but yet in whatever way the letters f and g, different from a and b, may be taken, nevertheless the situation always can be reduced to the previous, more simple [one]. To demonstrate this let us assume $x = \alpha + \beta z$ and our equation will become

$$\frac{(\alpha+f+\beta z)^n}{(\alpha+g+\beta z)^n} = \frac{a}{b} \quad \text{or} \quad \frac{(\frac{\alpha+f}{\beta}+z)^n}{(\frac{\alpha+g}{\beta}+z)^n} = \frac{a}{b};$$

and now it is clear that the quantities α and β always can be taken in such a way that it happens that

$$\frac{\alpha+f}{\beta} = a \quad \text{and} \quad \frac{\alpha+g}{\beta} = b,$$

[12]Translators: That is, the terms containing x^{n-1} cancel each other out.

[13]Translators: In other words, divide the previous equation by the coefficient of x^n, which is $a-b$, and then solve for x^n.

since indeed from this is deduced

$$\alpha = \frac{bf-ag}{a-b} \quad \text{and therefore} \quad \beta = \frac{f-g}{a-b}.$$

And thus that formula, which seemed much more general, can always be restored to that very simple one treated above nor then is anything new to be expected from it.

ANNOTATION ON THE EQUATIONS DEVELOPED ABOVE

14. If we should consider more carefully the forms that we have assigned for the roots of these equations above, all these things are discovered to conform excellently with that conjecture that I once[14] ventured to offer publicly, at the time when I asserted, for the solution of an equation of any degree, in which the second term is missing, as for instance

$$x^n = px^{n-2} + qx^{n-3} + rx^{n-4} + \text{etc.},$$

that there was always given a resolvent equation, lower by one degree, of this form

$$y^{n-1} - Ay^{n-2} + By^{n-3} - Cy^{n-4} + Dy^{n-5} - \text{etc.} = 0,$$

if the roots of which, numbering $n-1$, are α, β, γ, δ, ϵ etc., the result would be

$$x = \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \sqrt[n]{\epsilon} + \text{etc.}$$

15. Since therefore for the general form that we have treated above, a root has been found

$$x = \sqrt[n]{a^{n-1}b} + \sqrt[n]{a^{n-2}bb} + \sqrt[n]{a^{n-3}b^3} + \cdots + \sqrt[n]{ab^{n-1}},$$

from this it follows that the roots of the resolvent equation of order $n-1$ will be

$$a^{n-1}b, \quad a^{n-2}bb, \quad a^{n-3}b^3, \quad a^{n-4}b^4, \quad \ldots \quad ab^{n-1},$$

which therefore will be the values of that very y. Wherefore since the coefficient A is the sum of all these roots, it will be the case that

$$A = \frac{ab(a^{n-1} - b^{n-1})}{a-b};$$

moreover, the final member of this equation, in absolute value, will be the product of all these roots, which then will be

$$= a^{\frac{nn-n}{2}} b^{\frac{nn-n}{2}}.$$

For the rest of the terms let us run though the particular equations explained above.

I. For an equation of the third degree

[14] Stäckel points the reader to E30 and E282.

$$x^3 = 3abx + ab(a+b),$$

where the root was

$$x = \sqrt[3]{aab} + \sqrt[3]{abb}.$$

Here, if the resolvent equation should be assumed to be

$$yy - Ay + B = 0,$$

its roots will be aab and abb and therefore

$$A = ab(a+b)$$

and

$$B = a^3b^3.$$

II. For an equation of the fourth degree

$$x^4 = 6abxx + 4ab(a+b)x + ab(aa+ab+bb).$$

Here the root is

$$x = \sqrt[4]{a^3b} + \sqrt[4]{aabb} + \sqrt[4]{ab^3};$$

whence, if the resolvent equation should be assumed to be

$$y^3 - Ayy + By - C = 0,$$

its roots will be a^3b, $aabb$, ab^3, wherefore we shall have

$$\begin{aligned} A &= ab(aa+ab+bb), \\ B &= a^3b^3(aa+ab+bb) \qquad \text{and} \\ C &= a^6b^6. \end{aligned}$$

III. For an equation of the fifth degree

$$x^5 = 10abx^3 + 10ab(a+b)xx + 5ab(aa+ab+bb)x + ab(a^3+aab+abb+b^3).$$

Here therefore it will be the case that

$$x = \sqrt[5]{a^4b} + \sqrt[5]{a^3bb} + \sqrt[5]{aab^3} + \sqrt[5]{ab^4};$$

whence if the resolvent equation should be assumed to be

$$y^4 - Ay^3 + Byy - Cy + D = 0,$$

its roots will be a^4b, a^3bb, aab^3, ab^4, whence it is concluded that it will be the case that

$$\begin{aligned}
A &= ab(a^3 + aab + abb + b^3),\\
B &= a^3b^3(a^4 + a^3b + 2aabb + ab^3 + b^4),\\
C &= a^6b^b(a^3 + aab + abb + b^3),\\
D &= a^{10}b^{10}.
\end{aligned}$$

IV. For an equation of the sixth degree

$$\begin{aligned}
x^6 = {} & 15abx^4 + 20ab(a+b)x^3 + 15ab(aa + ab + bb)xx\\
& + 6ab(a^3 + aab + abb + b^3)x + ab(a^4 + a^3b + aabb + ab^3 + b^4).
\end{aligned}$$

Here then it will be held that

$$x = \sqrt[6]{a^5b} + \sqrt[6]{a^4bb} + \sqrt[6]{a^3b^3} + \sqrt[6]{aab^4} + \sqrt[6]{ab^5};$$

whence if the resolvent equation should be assumed to be

$$y^5 - Ay^4 + By^3 - Cyy + Dy - E = 0,$$

its roots will be a^5b, a^4bb, a^3b^3, aab^4, ab^5, whence it is concluded that it will be the case that

$$\begin{aligned}
A &= ab(a^4 + a^3b + aabb + ab^3 + b^4),\\
B &= a^3b^3(a^6 + a^5b + 2a^4bb + 2a^3b^3 + 2aab^4 + ab^5 + b^6),\\
C &= a^6b^6(a^6 + a^5b + 2a^4bb + 2a^3b^3 + 2aab^4 + ab^5 + b^6),\\
D &= a^{10}b^{10}(a^4 + a^3b + aabb + ab^3 + b^4),\\
E &= a^{15}b^{15},
\end{aligned}$$

where the middle formulae B and C can be more elegantly expressed in this way

$$\begin{aligned}
B &= a^3b^3(aa + bb)(a^4 + a^3b + aabb + ab^3 + b^4) \qquad \text{and}\\
C &= a^6b^6(aa + bb)(a^4 + a^3b + aabb + ab^3 + b^4),
\end{aligned}$$

which conclusions perhaps can shed some light on treating the general solution of equations with a more fortunate outcome.

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