On Establishing a Relationship Among Three or More Quantities[1]

Leonhard Euler
Translated by Geoffrey Smith
St. Mark's School

December 5, 2010

[1]Originally presented to the St. Petersburg Academy August 14, 1775 as "De relatione inter ternas pluresve quantitates instituenda." Enestrom index E591.

1. When two quantities A and B have been put forth, their relation or ratio is [able to be] defined, whenever two integers α and β can be located so that the equation $\alpha A = \beta B$ may be established and [the integers] be the smallest possible.[1] From this, if the quantities A and B should be commensurable, one may always assign those numbers α and β precisely, but if, however, they should be incommensurable, one may still give numbers α and β in such a way that that the difference between the expressions αA and βB is minimal, or [more precisely], so small that one cannot approach any nearer to equality between the expressions αA and βB , unless larger numbers should be used for α and β. And this problem of measures, originally proposed by Wallis, is readily solved where, when two numbers A and B, however large, are put forth, rationals are sought in smaller numbers so that they might express the relationship of those numbers as exactly as is able to be done if the larger numbers had not been employed.

2. In a similar manner, if three quantities A, B, and C should be put forth, three integers α β γ shall be able to be procured such that

$$\alpha A = \pm \beta B \mp \gamma C$$

is established; and indeed one will be able to allot all possible values for the numbers α, β, γ, from which (once they are found) one will with no difficulty at all produce the smallest numbers for α, β, and γ, and in this way the relation between the three quantities A, B, and C put forth is seen to be indicated most plainly. Moreover, the method of investigating the three numbers α, β, γ will be similar to that by which the relationship between two quantities is accustomed to be defined, and which is completed by operations of this kind, by which the greatest common divisor of two numbers is usually found. Let us illustrate in the following example.

3. Therefore, let the three following quantities be put forward: $A = 49$, $B = 59$, and $C = 75$, and let numbers a, b, c be sought such that $49a + 59b + 75c = 0$ is established, where a, b, c indicate integers, whether positive or negative[2]. Let that equation now be divided by the smallest of the quantities put forth, namely by 49, and let whatever numbers arise[3] be resolved into integral and fractional parts and the parts be put forth separately, so since the expressions taken side by side ought to equal zero. Let us establish the integral parts equal to the integer d and the fractional

[1] This definition of quantities with a 'defined ratio' is equivalent to the definition of commensurable numbers.

[2] Quidem left untranslated

[3] 'ex posterioribus terminis' left untranslated

part to the same number negated, $-d$. Thus, the two equations are produced

$$a + b + c = d$$

and

$$\frac{10b + 26c}{49} = -d.$$

Now from the latter equation is established $10b+26c+49d = 0$ which treated just like the first, through division by 10 will evidently give

$$b + 2c + 4d = 3e$$

and

$$\frac{6c + 9d}{10} = -3e.$$

Since these numbers 6 and 9 have the common divisor 3, we obviously immediately wrote $3e$ into the location of the simple letter e and thus the new equation will be $2c + 3d + 10e = 0$, which divided by 2 and distributed in a similar way gives these equations:

$$c + d + 5e = f$$

and

$$\frac{d}{2} = -f$$

the last of which immediately gives the equation $d = -2f$. Here the operations are ended, seeing that no more fractions remain.

4. Therefore, since it ought to be that $d = -2f$, and moreover since e has not been determined, the preceding terms will be defined by these two letters e and f going back in the following way:

$$c = 3f - 5e$$
$$b = 13e + 2f$$
$$a = -8e - 7f.$$

Therefore, the general solution of our problem, or the relation between the three numbers 49, 59, and 75 put forth, will be expressed by the following equation:

$$-(8e + 7f)49 + (13e + 2f)59 + (3f - 5e)75 = 0$$

where one may take whatever numbers for e and f.

5. Therefore we may see what kind of numbers for e and f make this equation most simple[4]. First, let us take $f = 1$ and $e = -1$, and the relationship found will be

$$1 \cdot 49 - 11 \cdot 59 + 8 \cdot 75 = 0,$$

but if we should take $e = 0$ and $f = 1$ the relationship will be

$$7 \cdot 49 - 2 \cdot 59 - 3 \cdot 75 = 0,$$

which without doubt is the simplest form of the relationship. And from this example it is now sufficiently clear that, however large the quantities A, B, and C may be, since we have arrived all the way to smaller divisors, at last all fractions are removed, and integers are always found instead of a, b, c.

6. Since, then, the matter is clear when the quantities A, B, C, and D put forth are rational, or if they are commensurable with each other, it is also evident that, if those quantities are irrational or even transcendental, then the operations used regularly here are never completed, and that therefore, such an exact relation can in no way be produced. Nevertheless, what must be noted from these first cases, if the operations which have been recounted should be broken off anywhere, then relations of this sort are going to be produced, which produce the thing not exactly, but nevertheless very close. This is a thing which will be able to be of use on many occasions, since among quantities of this sort the relation is only approximately true and, indeed, very little desired. However, let us demonstrate with several examples how one should handle the calculation in cases of this sort.[5]

7. Therefore, let there be three quantities $A = 1$, $B = \sqrt{2}$, $C = \sqrt{3}$, and first, so that the operations employed previously are able to find a solution, let us change these irrational quantities into decimal fractions, which we indeed do not continue beyond the sixth digit. It is truly:

$$\sqrt{2} = 1.414214$$

and

$$\sqrt{3} = 1.732051.$$

Now, through multiplication by 1000000 let this whole inquiry be recalled to integers, since this relation stands in rational numbers which these quantities

[4]Idiomatic translation: precise one sounds absolutely terrible

[5]S.6 translation due to Dr. Matthew Hartnett

have between themselves; in this manner the original equation $a+b\sqrt{2}+c\sqrt{3}$ will be changed into this

$$1000000a + 1414214b + 1732051c = 0$$

which, divided by 1000000 and distributed as above into two parts will give

$$a + b + c = +d$$

and

$$\frac{414214b + 732051c}{1000000} = -d.$$

The last reduced to integers therefore provides

$$414214b + 732051c + 1000000d = 0$$

and this equation, divided by 414214 and handled in the same way leads to equations

$$b + c + 2d = +e$$

and

$$\frac{317837c + 171572d}{414214} = -e.$$

The last of these[6] is brought to this state

$$317837c + 171572d + 414214e = 0.$$

This equation may be handled in the same way to yield these equations[7]

$$d + c + 2e = +f$$

and

$$\frac{146265c + 71070e}{171572} = -f.$$

The last of these when reduced to integers yields:

$$146265c + 71070e + 171572f = 0$$

which when divided by 71070 gives

$$e + 2c + 2f = +g$$

[6]Separated sentences for the sake of English readability

[7]Assuming prodeant is some Latin equivalent of middle provides the most reasonable translation.

and

$$\frac{4125c + 29432f}{71070} = -g.$$

The latter, having been reduced, becomes

$$4125c + 29432f + 71070g = 0$$

from which by being divided by 4125 these two equations are produced[8]:

$$c + 7f + 17g = +h$$

and

$$\frac{575f + 945g}{4125} = -h.$$

Moreover, the latter reduced to integers gives this:[9]

$$575f + 945g + 4125h = 0.$$

Now it is divided by 575 and will yield

$$f + g + 7h = +i$$

and

$$\frac{370g + 100h}{575} = -i,$$

or equivalently

$$\frac{74g + 20h}{115} = -i,$$

which, when reduced, becomes

$$74g + 20h + 115i = 0$$

from which through division by 20 these equations arise:

$$h + 3g + 5i = +k$$

and

$$\frac{14g + 15i}{20} = -k.$$

[8]Literally 'These two equations produce for themselves'

[9]Euler's arithmetic is faulty at this step(should have 557f instead of 575f; hence, through the end of S.8, his paper is false. However, since these sections mainly intended to explain Euler's methodology, rather than prove something, they are included as is.

8. One may in this way continue these operations however far it should please; but since the decimal fractions were not producded beyond the sixth figure, the last figures of our numbers are made continuously more uncertain through these operations, from where in the last equation one may see the two numbers 14 and 15 to be essentially equal between themselves, from which it will be able to be found that $g = 1$ and $i = -1$.[10] Therefore $k = 0$ will be established, and hence the following values will be found by going backwards:

$$\begin{aligned} h &= 2 \\ f &= 16 \\ c &= 97 \\ e &= -161 \\ d &= 209 \\ b &= -676 \\ a &= 788. \end{aligned}$$

Thus the sought-after relation will have this:

$$788 - 676\sqrt{2} + 97\sqrt{3} = 0$$

or

$$676\sqrt{2} - 97\sqrt{3} = 788$$

for which the error rises up scarcely beyond the sixth decimal figure.

9. Moreover, whatever degree of accuracy this relation should approach, nevertheless from there one may by no means conclude that it is exact. For if with a, b, c designating rational numbers, it were true that $a = b\sqrt{2} + c\sqrt{3}$, then squaring would yield $a^2 = 2b^2 + 3c^2 + 2bc\sqrt{6}$. Hence, $\sqrt{6} = \frac{a^2-2b^2-3c^2}{2bc}$, and therefore $\sqrt{6}$ would be a rational number, which would be absurd. This same thing must be maintained from all other numbers with roots of whatever order in such a way that any irrational quantity by its nature is so greatly distinct from all other irrationals not only of similar but also of distinct degrees that clearly no rational relation can be found between more irrational numbers of this sort.

10.However, whether transcendental quantities, such as those which involve the circumfrence of a circle or logarithms[11], are able to be compared

[10] This is a bit of mathematical trickery Euler uses to attain a reasonable result despite his error above.

[11] At the time of this paper's presentation (1775), the existence of transcendental numbers had been conjectured, but not proven.

even with any sort of algebraic quantity, seems at this point most uncertain, if indeed such an impossibility has been shown by no one. Indeed, it seems demonstrated so greatly enough that the perimeter of a circle with diameter 1 allows no comparison with simple quadratic radical formulas, since otherwise, a continued fraction, equal to π itself, ought to have periodic terms, a thing which does not seem to happen at all. We are compelled to leave in doubt whether the quantity π is able to be compared with formulas arranged such in absolutely no way. Therefore, let us begin such an investigation for the relationship of the quantities π, $\sqrt{2}$, and $\sqrt{3}$ determined by the method just set forth.

11. Let us then investigate this equation with the methodology explained:

$$a\sqrt{2} + b\sqrt{3} + c\pi = 0$$

which is expressed in integers approximately thus:

$$1414214a + 1732051b + 3141593c = 0.$$

This, when divided by the smallest number, provides these equations:

$$a + b + 2c = +d$$

and

$$\frac{317837b + 313165c}{1414214} = -d.$$

The last equation accordingly is made into integers

$$317837b + 313165c + 1414214d = 0,$$

which again may be divided by the smallest number, by which action these two equations come forth:

$$c + b + 4d = +e$$

and

$$\frac{4672b + 161554d}{313165} = -e.$$

The latter of these, when reduced, becomes:

$$4672b + 161554d + 313165e = 0.$$

Through division by 4672 these equations are garnered:

$$b + 34d + 67e = +f$$

and

$$\frac{2706d + 141e}{4672} = -f.$$

12. I will not continue these operations further, since if an exact relation were to be given, without doubt it would not be so complicated. Moreover it would have been of little use to provide such nearly true relations. From this point, the idea seems to be certain enough, because the perimeter of a circle should establishe such a type of transcendental quantities that it may allow itself in no way to be compared with any other quantities, whether surds or transcendentals of another type.

13. Moreover, there exist other infinite classes of transcendentals, which are not able to be reduced either to a circle nor to logarithms, even if they should seem to hold any relation with those numbers. If by chance such quantities were to have any exact relation with some hitherto known number [12], which one may not define directly from analytic principles, this method seems to supply a unique way by which one, with a benefit just as intuition, may investigate relations of this sort.

14. Therefore, I will fairly accurately present here a single example of this sort for which such a relationship might exist. Consider the sum of a reciprocal series of cubes:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \dots,$$

which I was hitherto able by no means to reduce, whether to a circle or to logarithms, although neverthelesss the sum of all the second powers is able to be produced through second powers of π itself, and moreover the alternating sum of the first powers

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

expresses the logarithm of 2.

15. Therefore, since this reciprocal series of cubes

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \dots,$$

contains in itself the cube of this series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

[12] Quandum translated as quandam

it seems probable that in this sum the value $(\ln(2))^3$ ought to occur, and nevertheless it is not certain that the sum is equal to any multiple of this quantity. Then, since the same series contains in itself the product from the two lower-order ones, namely:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - etc = \ln(2)$$

and

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + etc = \frac{\pi^2}{6}$$

one may conjecture that in the same manner that this product $\frac{\pi^2}{6}\ln(2)$ also occurs. For this reason it will be worth the effort to inquire whether by chance the sum of the reciprocals of cubes is made equal to a formula so put forth

$$\alpha(\ln(2))^3 + \beta\frac{\pi^2}{6}\ln(2)$$

in such a way that α and β are rational numbers.

16. Back in the day I assigned, through approximations, the sum of the reciprocal series of cubes as such:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + etc = 1.202056903,$$

from which if its fourth part should be subtracted, the sum of this series produces

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} etc = .901542677.$$

For the sake of brevity let us set that equal to A, and let us seek numbers a, b, c such that

$$aA + b(\ln(2))^3 + c\ln(2)\frac{\pi^2}{6} = 0.$$

Since

$$\frac{\pi^2}{6} = 1.644923066$$

and

$$\ln(2) = .693147180,$$

we conclude that there will most nearly be:

$$(\ln(2))^3 = .333025$$

and

$$\ln(2)\frac{\pi^2}{6} = 1.140182.$$

From this must appear the relationship:

$$901543a + 333025b + 1140182c = 0.$$

17.Thus, in place of the equation, operations may be used as above, and there will be through division by 333025

$$b + 2a + 3c = +d$$

and

$$\frac{235493a + 141107c}{333025} = -d.$$

Moreover, the latter reduced to integers yields

$$235493a + 141107c + 333025d = 0$$

from which through division by 141107 we deduce these equations

$$a + c + 2d = +e$$

and

$$\frac{94386a + 50811d}{141107} = -e$$

or

$$94386a + 50811d + 141107e = 0.$$

From this equation through division by 50811 we conclude[13]

$$a + d + 2e = +f$$

and

$$\frac{43575a + 39485e}{50811} = -f.$$

Moreover this latter equation reduced to integers establishes

$$43575a + 39485e + 50811f = 0$$

from which by further division by the smallest number these equations arise:

$$a + e + f = +g$$

[13]Colligitur is passive, but the active form is far more typical in English.

and

$$\frac{4090a + 11326f}{39485} = -g$$

or

$$4090a + 11326f + 39485g = 0$$

from where these relations may be formed:

$$a + 2f + 9g = +h$$

and

$$\frac{3146f + 2675g}{4090} = -h.$$

And so on.

18. It would be superfluous to continue these operations further, since one may now understand suffficiently well from here, that no relationship between the three quantities taken is given that is able to be resolved consistent with the truth. Therefore, since I tried in vain to explore the investigation of this reciprocal sum of cubes in different ways and this method was called into use without result, it seems from investigation that it rightly must be abandoned.

Acknowledgements

There are a number of people whose assistance was instrumental in making this translation possible. Dr. Matthew Hartnett, in addition to personally translating section 6, originally proposed the idea of an Euler translation and helped edit the translation for faithfulness. David Conti provided further assistance with editing. Dr. James Tanton ensured the mathematical consistency of the final translation. Finally, Dr. William Dunham inspired the idea of translating Euler and provided a remarkably legible copy of the paper in Latin from the *Opera Omnia* to augment the original paper.