On three square numbers, of which the sum and the sum of products two apiece will be a square[i]

by Leonhard Euler

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§. 1.

In Volume VIII of the *Novi Commentarii*, I treated a problem in which three numbers are examined whose sum is a square, just as the sum of their products two at a time (is a square), along with the product of all of them becomes a square, the solution of which, since it was not only very difficult but also gave rise to immense numbers, can be clearly seen, if in addition a ninth[ii] condition is added, the solution would completely overwhelm the abilities of the Mathematicians. But it turns out in the investigation, which I will treat here, in addition to three aforementioned conditions, even this is necessary: that the numbers taken one by one be squared. But even so, with this condition added, after some futile attempts, at last I found a method of solving such a problem sufficiently enough where finally it is possible to identify numbers sufficiently small which satisfy the problem.

§. 2. Let three squared numbers be considered: xx, yy, zz; such that they should satisfy,

I. $xx + yy + zz = \square$.

II. $xxyy + xxzz + yyzz = \square$,

of these requirements the first will be met by taking

$$x = pp + qq - rr; \ y = 2pr \text{ and } z = 2qr;$$

when it turns out,

$$xx + yy + zz = (pp + qq + rr)^2,$$

[i] L. Euler, *De Tribus Numeris Quadratis, quorum tam summa, quam summa productorum ex binis sit quadratum* (E523). Acta *Academiae Scientarum Imperialis Petropolitinae* 1779, 1782, pp. 30-39. Reprinted in *Commentat. arithm.* 2, 1849, pp. 457-461.

[ii] Ed. suggest *nova* for *nona*, “a new condition,” due to a typesetting error in the original document.

from which, if we suppose $xx + yy + zz = P^2$, with the assumptions

$$x = pp + qq - rr,\ y = 2pr,\ z = 2qr,\ \text{making } P = pp + qq + rr.$$

§. 3. Let us now proceed to the second requirement which demands that

$$xx(yy + zz) + zzyy = Q^2;$$

which therefore becomes

$$yy + zz = 4rr(pp + qq),$$

from which this equation will arise:

$$Q^2 = 4rr(pp + qq)(pp + qq - rr)^2 + 16ppqqr^4,$$

which, when divided by the square factor $4rr$, will give

$$\frac{QQ}{4rr} = (pp + qq)(pp + qq - rr)^2 + 4ppqqrr,$$

which in turn must be rendered as a square. However by working this out the letters p and q will be raised to the sixth power, (and) r to the fourth power, which therefore seems to be able to be pursued sufficiently, since indeed the case is clear on its own, naturally if $rr = pp + qq$, as long as $pp + qq$ is a square. But from this point it is not possible to derive even one other solution; therefore the task must be approached by another method which will be shown by the following, indeed extraordinary, approach.

§. 4. I set $r = p - nq$, so that by this method, no restriction is inferred, since in the place of the letter r a new variable n is introduced; then our equation will take on this form:

$$\frac{QQ}{4(p - nq)^2} = (pp + qq)(2npq + (1 - nn)qq)^2 - 4ppqq(p - nq)^2,$$ (iii)

which now can be divided by qq, such that

$$\frac{QQ}{4qq(p - nq)^2} = (pp + qq)(2np + (1 - nn)q)^2 - 4pp(p - nq)^2,$$ (iv)

(iii)Ed: The last term in this equation should be positive, yielding $\frac{QQ}{4(p-nq)^2} = (pp+qq)(2npq+(1-nn)qq)^2 + 4ppqq(p - nq)^2$.

(iv)Ed: Again, the last term in this equation should be positive, yielding $\frac{QQ}{4qq(p-nq)^2} = (pp + qq)(2np + (1 - nn)q)^2 + 4pp(p - nq)^2$.

becomes a square, which for the sake of brevity, we will refer to as R^2, so that in turn $Q = 2q(p - nq)R$. Now, when this expansion has been done, this equation will produce:

$$R^2 = 4(1+nn)p^4 - 4n(1+nn)p^3q + (1+6nn+n^4)ppqq + 4n(1-nn)pq^3 + (1-nn)^2q^4,$$

in this formula the final term has become a square; the first term ought to be rendered a square, by making $nn + 1 = \square$; but in truth it is sufficient for the method, that the final term be a square.

§. 5. For R^2 let us propose a square of this sort, whereby the three final terms be taken away, and from the two remaining terms the ratio between p and q should be determined. To this end, let it be proposed that

$$R = (1+nn)qq + 2npq + \alpha pp, ^{(\text{v})}$$

and α should be determined, in such a way that even the third-to-last term cancels, which happens by taking $\alpha = \frac{1+2nn+n^4}{2(1-nn)}$, when this is done, the remaining equation will be:

$$4p^4 - 4np^3q = \frac{(1+nn)^3}{4(1-nn)^2}p^4 + \frac{2n(1+nn)}{1-nn}p^3q,$$

if then, by multiplying by $4(1-nn)^2$, by dividing by p^3, and by transferring the letters p and q to the same side, will become

$$(15 - 35nn + 13n^4 - n^6)p = 8n(1-nn)(3-nn)q,$$

this equation next can be divided by $3-nn$, and when this is done, becomes $(5-10nn+n^4)p = 8n(1-nn)q$, from which is deduced,

$$\frac{p}{q} = \frac{8n(1-nn)}{5-10nn12+n^4}.^{(\text{vi})}$$

§. 6. Therefore let us set $q = 5 - 10nn + n^4$ and $p = 8n(1-nn)$ so that it satisfies this equation, from these values it is deduced that

$$r = p - nq = n(3 + 2nn - n^4).$$

In addition, with these values substituted we find

$$R = (1-nn)((5-10nn+n^4)^2 + 16nn(5-10nn+n^4) + 32(1+nn)^2).$$

(v) Ed. Should be $R = (1-nn)qq + 2npq + \alpha pp$.

(vi) Ed: The denominator should be $5 - 10nn + n^4$.

However, with these values discovered, these very numbers examined will thus be formed such that

$$x = pp + qq - rr;\;\; y = 2pr;\;\; z = 2qr.$$

With the aid of these formulas, therefore we will produce some examples.

Example I.

§. 7. Let $n = 2$, and $p = -48$; $q = -19$; $r = 10$,[(vii)] making $R = 7035$. And so,

$$Q = 2qrR = 4 \cdot 5^2 \cdot 19 \cdot 1407.$$

From this then these numbers will yield:

$$x = 2565;\;\; y = 2 \cdot 10 \cdot 48;\;\; z = 2 \cdot 10 \cdot 19.$$

But since these numbers have a common divisor of 5, through this divisor they can be reduced, similarly the number P will turn out five times less, but Q will be twenty-five times less, and in this way the solution will be comprised of the following values:

$$P = 553;\;\; Q = 106932;\;\; x = 513;\;\; y = 192;\;\; z = 76.$$

Now the squares of the numbers x, y, z will be numbers of the sort, that will satisfy the aforementioned Problem. Such numbers will therefore be,

$$x^2 = 263169;\;\; y^2 = 36864;\;\; z^2 = 5776,$$

which are numbers amazingly smaller than those which I derived in the article cited above; from which it can be understood, that the method I used at that time was not sufficient. The sum of these three numbers is $= 553^2$; the sum of the product taken two at a time is $= 35948^2$ and the product of them all is $= 513^2 \cdot 192^2 \cdot 76^2$.

Example II.

§. 8. Let $n = 3$, and

$$p = -8 \cdot 24 = -192;\;\; q = -4;\;\; r = -180,$$

[(vii)] Ed. Should be $r = -10$.

these numbers will be reduced by a factor of -4

$$p = 48; q = 1; r = 45; \text{ and thus}$$

$$R = 14120, \text{ from this } Q = 18 \cdot 25 \cdot 2824.$$

From this point the examined numbers will yield

$$x = 280; y = 90 \cdot 48; z = 90,$$

and if reduced by a factor of 10 it turns out that

$$x = 28; y = 432; z = 9; P = 433; Q = 12708,$$

which numbers are still less than the preceding ones, and for that reason seem to be the smallest of them all which satisfy the problem. Thus the squares of these numbers, which are

$$x^2 = 784; y^2 = 186624; z^2 = 81,$$

will be without a doubt the smallest ones satisfying the Problem above, naturally the sum is 433^2; the sum of the squares two at a time is $= 12708^2$ and the product of them all is $28^2 \cdot 432^2 \cdot 9^2$.

Example III.

§9. Let $n = \frac{1}{2}$, and $p = 3$; $q = \frac{41}{16}$, $r = \frac{55}{32}$, or if, with all these numbers multiplied by 32, it yields $p = 9$(viii); $q = 82$; $r = 55$; which will yield

$$R = 22515 \text{ and } Q = 2 \cdot 82 \cdot 55 \cdot 22515.$$

Then it turns out that

$$x = 12915; y = 2 \cdot 55 \cdot 96; z = 2 \cdot 55 \cdot 82,$$

these numbers can be reduced by a factor of 5, which when done yields

$$x = 2583; y = 2112; z = 1804, \text{ or}$$

$$x = 3 \cdot 7 \cdot 123; y = 3 \cdot 11 \cdot 64; z = 4 \cdot 11 \cdot 41,$$

(viii)Ed: Should be 96.

§10. All these things have been deduced from the first solution of the two-square formula introduced in §4. However, there does exist a method by which, from some solution already found, more new solutions can be derived, but in this way it arrives at formulas far too complicated; a task which I am not undertaking here; especially since in such investigations one is usually directed that simpler solutions at least be drawn out.

Test Cases.

in which $nn + 1$ is a square.

§11. Therefore let $nn + 1 = mm$, because it turns out, that when $n = \frac{aa-bb}{2ab}$; then $m = \frac{aa+bb}{2ab}$, after this observation, let us retain in the formula the letters m and n, and the equation must be resolved as,

$$R^2 = 4mmp^4 - 4nmmp^3q + (m^4 + 4nn)ppqq + 4n(1 - nn)pq^3 + (1 - nn)^2q^4,$$

where, now the first, just as last term has been squared, and for that reason, except for the preceding operation, the three first terms will be arranged for consideration, which therefore we will pursue in order.

§12. In the first case therefore, let

Operation I. $$R = 2mpp - nmpq + (1 - nn)qq$$

where it should be noted that the number m can be taken as a positive or a negative, from which therefore two solutions will arise. Of this value for R squared, with the previous expression above with R^2 subtracted, the following equation will arise:

$$\frac{p}{q} = \frac{4m + 2mn - 2mn^3 - 4n^3}{4m - 4mnn + mmnn - 4nn - m^4}$$

and if $nn = mm - 1$,

$$\frac{p}{q} = \frac{2n(4 + 2m - 2mm - m^3)}{4 + 8m - 5mm - 4m^3}.$$

§13. Since the letters m and n always are fractions, so that they may be reduced more easily, let us introduce the variable factor Δ and let us set

$$p = 2\Delta n(4 + 2m - 2mm - m^3) \text{ and}$$

$$q = \Delta(4 + 8m - 5mm - 4m^3),$$

from which, because $r = p - nq$, there will be

$$r = \Delta n(4 - 4m + mm + 2m^3).$$

§14. Consequently, with these three values discovered, the numbers x, y, z under discussion are thus determined, so that

$$x = pp + qq - rr;\ y = 2pr;\ z = 2qr.$$

Further, however, there will be

$$P = pp - qq + rr;\ Q = 2qrR,$$

of course, then

$$R = 2mpp - mnpq + (1 - nn)qq.$$

§15. Let us try a unique example, so that it might be clear, whether or not from here smaller numbers than those earlier can be discovered. And so let us set $a = 2$ and $b = 1$, so that

$$n = \tfrac{3}{4} \text{ and } m = \pm\tfrac{5}{4}, \text{ and from this will arise}$$

$$p = \tfrac{3}{2}\Delta(4 \pm \tfrac{5}{2} - \tfrac{25}{8} \mp \tfrac{125}{64}),\ q = \Delta(4 \pm 10 - \tfrac{125}{16} \mp \tfrac{125}{16})$$

or

$$p = \tfrac{3}{2}\Delta(\tfrac{7}{8} \pm \tfrac{35}{64}) \text{ and } q = \Delta(-\tfrac{61}{16} \pm \tfrac{35}{16}).$$

Let us set $\Delta = 128$, and there will be

$$p = 3(56 \pm 35)\ q = 8(-61 \pm 35),$$

and from here

$$r = p - \tfrac{3}{4}q = 3(178 \mp 35).^{(\mathrm{ix})}$$

Taking the upper sign, since in this case the resulting numbers can be divided by 13, by which, it is discovered that:

$$p = 3 \cdot 7 = 21; q = -2 \cdot 8 = -16; r = 3 \cdot 11 = 33,$$

from which we acquire:

$$x = -392; y = 1386; z = -1056,$$

which, once again, when the signs are removed, and are divided by 2, such that

$$x = 196; y = 693; z = 528.$$

(ix) Ed: Final) inserted by translator.

However, we have already found numbers much smaller.

Operation II. §16. So that, except for the first term, two final terms might cancel, let us set:

$$R = 2mpp + 2npq + (1 - nn)qq;$$

from which will arise the following equation:

$$\frac{p}{q} = \frac{4m - 4mnn - m^4}{4mn(2+m)}, \text{ or}$$

$$\frac{p}{q} = \frac{8 - 4mm - m^3}{4n(2+m)}, \text{ because } nn = mm - 1,$$

or even

$$\frac{p}{q} = \frac{(2+m)(4 - 2m - mm)}{4n(2+m)} = \frac{4 - 2m - mm}{4n}.$$

Let there be as above

$$p = \Delta(4 - 2m - mm) \text{ and } q = 4\Delta n$$

and from here will arise

$$r = p - nq = \Delta(8 - 2m - 5mm);$$

finally

$$x = pp + qq - rr; y = 2pr; z = 2qr;$$

$$P = pp + qq + rr \text{ and } Q = 2qrR,$$

of course, then

$$R = 2mpp + 2npq + (1 - nn)qq.$$

§17. So that the matter might be illustrated by an example, again let us set $n = \frac{3}{4}$, and likewise $m = \pm\frac{5}{4}$, and there will be,

$$p = \Delta(\tfrac{39}{16} \mp \tfrac{5}{2})^{(x)} \text{ and } q = 3\Delta.$$

Let $\Delta = 16$, and, taking the upper sign, there will be $p = -1$ and $q = 48$; from there then $r = -37$,(xi), from which the desired numbers become

$$x = 936; y = 74; z = 3552, ^{(xii)}$$

(x)Ed: This should be $p = \Delta(-\frac{39}{16} \pm \frac{5}{2})$.

(xi)Ed: The numbers in this sentence should be $p = 1$, $q = 48$, and $r = -35$.

(xii)Ed: Should be $x = 1080$, $y = -70$, and $z = -3360$.

or by dividing,

$$x = 468; y = 37; z = 1776,^{\text{(xiii)}}$$

which are still larger than the preceding ones.[xiv]

§18. Operation III. Let us now subtract three earlier terms by setting

$$R = 2mpp - mnpq + \tfrac{m^4+3nn}{4m}qq,$$

from which this equation follows:

$$g\left(\tfrac{(m^4+3nn)^2}{16mm} - (1-nn)^2\right) = -\tfrac{n}{2}(m^4 - 5nn + 8)p.^{\text{(xv)}}$$

From this form it is clear enough that no smaller numbers which satisfy the problem, can be extracted, and for this reason, we refrain from any further attempt.

[xiii] Ed: Should be $x = 108$, $y = 7$, and $z = 336$.

[xiv] Ed: Interestingly, $x^2 + y^2 + z^2 = 353^2$, which is smaller than the total in Example II, but $x^2y^2 + x^2z^2 + y^2z^2 = 36,372^2$ which is larger than the total in Example II.

[xv] Ed: Should be $q(m^8 - 2m^4n^2(n^2-3) + n^4(n^2-3)^2 - 16m^2(n^2-1)^2) = 8m^2n(8 + m^4 - 5n^2 - n^4)p$.

De

TRIBUS NUMERIS QUADRATIS,

QUORUM TAM SUMMA, QUAM SUMMA

PRODUCTORUM EX BINIS SIT

QUADRATUM.

Auctore

L. EULERO

§. 1.

In Tomo Novorum Commentariorum VIII. tractavi Problema, quo tres numeri quaeruntur, quorum tam summa, quam summa productorum ex binis, una cum producto omnium fiant quadrata, cuius Solutio cum non solum esset difficillima, sed etiam ad immensos numeros perduxisset, merito videri poterat, si insuper nona[xvi] conditio adderetur, solutionem vires Analyieos penitus esse superaturam. Hoc tamen evenit in quaestione, quam hic tractabo, ubi praeter tres conditiones memoratas etiam haec postulatur; ut singuli numeri quaesiti sint quadrati. Interim tamen hac conditione adiecta, post plures conatus irritos, tandem modum inveni istud Problema satis commode resolvendi, ubi adeo numeros satis modicos assignare licet Problemati satisfacientes.

§. 2. Sint xx, yy, zz, terni numeri quadrati quaesiti; ita ut esse debeat,

I. $xx + yy + zz = \Box$.

II. $xxyy + xxzz + yyzz = \Box$,

quarum conditionum priori satisfiet, sumendo

$$x = pp + qq - rr;\ y = 2pr \text{ et } z = 2qr;$$

cum enim erit,

$$xx + yy + zz = (pp + qq + rr)^2,$$

unde si ponamus $xx + yy + zz = P^2$, sumtis

$$x = pp + qq - rr,\ y = 2pr,\ z = 2qr,\ \text{fiet } P = pp + qq + rr.$$

[xvi] Ed. suggest *nova* for *nona*, "a new condition," due to a typesetting error in the original document.

§. 3. Prodrediamur nunc ad alteram conditionem, quae postulat, ut fit

$$xx(yy+zz)+zzyy=Q^2;$$

quare cum fit

$$yy+zz=4rr(pp+qq),$$

hinc orietur ista aequatio:

$$Q^2=4rr(pp+qq)(pp+qq-rr)^2+16ppqqr^4,$$

quae divisa per factorem quadratum $4rr$ dabit

$$\frac{QQ}{4rr}=(pp+qq)(pp+qq-rr)^2+4ppqqrr,$$

quam ergo formulam quadratum reddi oportet. Ea autem evoluta literae p et q ad sextam potestatem ascendent, litera vero r tantum ad quartam, quae ergo commode investigari posse videtur, siquidem casus sponte patet, scilicet si $rr=pp+qq$, dummodo fuerit $pp+qq$ quadratum. Interim tamen hinc ne unicam quidem aliam solutionem derivare licet; unde negotium prorsus alio modo aggredi oportet, quod sequenti modo egregio successu praestari poterit.

§. 4. Pono autem $r=p-nq$, ita ut hoc modo nulla restrictio inferatur, quoniam loco literae r nova indeterminata n introducitur; tum autem nostra aequatio hanc induet formam:

$$\frac{QQ}{4(p-nq)^2}=(pp+qq)(2npq+(1-nn)qq)^2-4ppqq(p-nq)^2,$$

quae iam dividi potest per qq, ita ut

$$\frac{QQ}{4qq(p-nq)^2}=(pp+qq)(2np+(1-nn)q)^2-4pp(p-nq)^2,$$

quod quadratum brevitatis gratia designemus per R^2, ita ut fit $Q=2q(p-nq)R$. Nunc igitur facta evolutione prodibit haec aequatio:

$$R^2=4(1+nn)p^4-4n(1+nn)p^3q+(1+6nn+n^4)ppqq+4n(1-nn)pq^3+(1-nn)^2q^4,$$

in qua formula postremum membrum evasit quadratum; primum vero membrum reddi posset quadratum, faciendo $nn+1=\square$; at vero ad solutionem sufficere potest, ut postremus tantum terminus sit quadratum.

§. 5. Pro R^2 eiusmodi quadratum statuamus, quo sublato tres ultimi termini e medio tollantur, et ex duobus prioribus relictis ratio inter p et q determinetur. Hunc in finem statuatur

$$R = (1+nn)qq + 2npq + \alpha pp,$$

et α ita determinetur, ut etiam antepenultimus aufferatur, quod sit sumendo $\alpha = \frac{1+2nn+n^4}{2(1-nn)}$, quo facto aequatio relicta erit:

$$4p^4 - 4np^3q = \frac{(1+nn)^3}{4(1-nn)^2}p^4 + \frac{2n(1+nn)}{1-nn}p^3q,$$

sive per $4(1-nn)^2$ multiplicando, per p^3 dividendo et literas p et q ad eandem partem transferendo fiet,

$$(15 - 35nn + 13n^4 - n^6)p = 8n(1-nn)(3-nn)q,$$

quae aequatio porro per $3-nn$ dividi potest, quo facto fit $(5-10nn+n^4)p = 8n(1-nn)q$, unde deducitur,

$$\frac{p}{q} = \frac{8n(1-nn)}{5-10nn12+n^4}.$$

§. 6. Sumamus igitur, ut huic aequationi satisfiat, $q = 5-10nn+n^4$ et $p = 8n(1-nn)$, ex quibus valoribus colligitur

$$r = p - nq = n(3+2nn-n^4).$$

Praeterea vero his valoribus substitutis invenimus

$$R = (1-nn)((5-10nn+n^4)^2 + 16nn(5-10nn+n^4) + 32(1+nn)^2).$$

Inventis autem his valoribus ipsi numeri quaesiti ita formabuntur, ut fit

$$x = pp + qq - rr; \; y = 2pr; \; z = 2qr.$$

Ope harum formularum igitur aliquot exempla evolvamus.

Exemplum I.

§. 7. Sit $n = 2$, eritque $p = -48$; $q = -19$; $r = 10$, unde fit $R = 7035$. Erat autem,

$$Q = 2qrR = 4 \cdot 5^2 \cdot 19 \cdot 1407.$$

Hinc vero ipsi numeri quaesiti ita se habebunt:

$$x = 2565;\; y = 2 \cdot 10 \cdot 48;\; z = 2 \cdot 10 \cdot 19.$$

Quoniam autem hi numeri communem divisorem habent 5, per eius divisionem deprimi poterunt, simulque numerus P quinquies evadet minor, at vero Q vicies quinquies minor, hocque modo solutio sequentibus valoribus continebitur:

$$P = 553;\; Q = 106932;\; x = 513;\; y = 192;\; z = 76.$$

Quadrata iam numerorum x, y, z eiusmodi erunt numeri, qui Problemati olim tractato satisfacient. Tales igitur numeri erunt,

$$x^2 = 263169;\; y^2 = 36864;\; z^2 = 5776,$$

qui numeri sunt incomparabiliter minores iis, quos loco citato exhibui; unde intelligitur, methodum, qua tum temporis sum usus, non satis esse accommodatam. Summa autem horum trium numerorum est $= 553^2$; summa productorum ex binis $= 35948^2$ et productum omnium $= 513^2 \cdot 192^2 \cdot 76^2$.

Exemplum II.

§. 8. Sit $n = 3$, eritque

$$p = -8 \cdot 24 = -192;\; q = -4;\; r = -180,$$

qui numeri per -4 depressi evadent

$$p = 48; q = 1; r = 45;\; \text{unde fit}$$

$$R = 14120,\; \text{hincque } Q = 18 \cdot 25 \cdot 2824.$$

Hinc vero ipsi numeri quaesiti erunt

$$x = 280; y = 90 \cdot 48; z = 90,$$

sive deprimendo per 10 fiet

$$x = 28; y = 432; z = 9; P = 433; Q = 12708,$$

qui numeri adhuc praecedentibus sunt minores, ideoque minimi omnium esse videntur qui satisfaciant. Quadrata ergo horum numerorum, quae sunt

$$x^2 = 784; y^2 = 186624; z^2 = 81,$$

erunt sine dubio minimi Problemati olim tractato satisfacientes, quippe quorum summa est 433^2; summa quadratorum ex binis $= 12708^2$ et productum omnium $28^2 \cdot 432^2 \cdot 9^2$.

Exemplum III.

§9. Sit $n = \frac{1}{2}$, fietque $p = 3$; $q = \frac{41}{16}$, $r = \frac{55}{32}$, sive, ductis his omnibus numeris in 32, fiet $p = 9$; $q = 82$; $r = 55$; unde fit

$$R = 22515 \text{ et } Q = 2 \cdot 82 \cdot 55 \cdot 22515.$$

Tum vero erit

$$x = 12915; y = 2 \cdot 55 \cdot 96; z = 2 \cdot 55 \cdot 82,$$

qui numeri per 5 deprimi possunt, quo facto fit

$$x = 2583; y = 2112; z = 1804, \text{ sive}$$

$$x = 3 \cdot 7 \cdot 123; y = 3 \cdot 11 \cdot 64; z = 4 \cdot 11 \cdot 41,$$

§10. Haec omnia ex formulae biquadraticae §4 allatae prima resolutione sunt deducta. Constat autem methodus, qua ex qualibet resolutione iam inventa plures novae derivari possunt; verum hoc modo ad formulas nimis complicatas perveniretur, quod negotium hic non suscipio: praecipue enim in talibus investigationibus is solet intendi, ut solutiones saltem simpliciores eruantur.

Evolutio casuum.

quibus est $nn + 1$ quadratum.

§11. Sit igitur $nn + 1 = mm$, quod evenit, quoties fuerit $n = \frac{aa-bb}{2ab}$; tum enim erit $m = \frac{aa+bb}{2ab}$, quo observato retineamus in calculo literas m et n, eritque aequatio resolvenda,

$$R^2 = 4mmp^4 - 4nmmp^3q + (m^4 + 4nn)ppqq + 4n(1 - nn)pq^3 + (1 - nn)^2q^4,$$

ubi iam tam primus quam ultimus terminus sunt quadrata, ideoque praeter operationem praecedentem tres adhuc respectu primi termini institui poterunt, quas ergo ordine prosequemur.

§12. Primo igitur ponatur

Operat. I. $$R = 2mpp - nmpq + (1 - nn)qq$$

ubi notetur, numerum m tam positive quam negative accipi posse, unde ergo gemina solutio nascetur. Huius ergo valoris pro R quadrato a superiore expressione pro R^2 sublato orietur sequens aequatio:

$$\frac{p}{q} = \frac{4m + 2mn - 2mn^3 - 4n^3}{4m - 4mnn + mmnn - 4nn - m^4}$$

sive ob $nn = mm - 1$ erit

$$\frac{p}{q} = \frac{2n(4 + 2m - 2mm - m^3)}{4 + 8m - 5mm - 4m^3}.$$

§13. Quoniam literae m et n semper sunt fractiones, quo eae facilius tollantur, introducamus multiplicatorem indefinitum Δ ponamusque

$$p = 2\Delta n(4 + 2m - 2mm - m^3) \text{ et}$$

$$q = \Delta(4 + 8m - 5mm - 4m^3),$$

unde ob $r = p - nq$ fiet

$$r = \Delta n(4 - 4m + mm + 2m^3).$$

§14. His igitur tribus valoribus inventis numeri quaesiti x, y, z ita ex iis determinantur, ut sit

$$x = pp + qq - rr;\ y = 2pr;\ z = 2qr.$$

Praeterea vero erit

$$P = pp - qq + rr;\ Q = 2qrR,$$

existente

$$R = 2mpp - mnpq + (1 - nn)qq.$$

§15. Unicum exemplum evolvamus, ut pateat, num hinc minores numeri sint prodituri quam ante. Sumamus igitur $a = 2$ et $b = 1$, fietque

$$n = \tfrac{3}{4} \text{ et } m = \pm\tfrac{5}{4},\ \text{hincque fiet}$$

$$p = \tfrac{3}{2}\Delta(4 \pm \tfrac{5}{2} - \tfrac{25}{8} \mp \tfrac{125}{64}),\ q = \Delta(4 \pm 10 - \tfrac{125}{16} \mp \tfrac{125}{16})$$

sive

$$p = \tfrac{3}{2}\Delta(\tfrac{7}{8} \pm \tfrac{35}{64}) \text{ et } q = \Delta(-\tfrac{61}{16} \pm \tfrac{35}{16}).$$

Sumamus $\Delta = 128$, fietque

$$p = 3(56 \pm 35)\ q = 8(-61 \pm 35),$$

hincque

$$r = p - \tfrac{3}{4}q = 3(178 \mp 35).^{\text{(xvii)}}$$

Valeat signum superius, quoniam hoc casu numeri resultantes per 13 deprimi possunt, quo facto reperitur:

$$p = 3 \cdot 7 = 21; q = -2 \cdot 8 = -16; r = 3 \cdot 11 = 33,$$

unde colligimus:

$$x = -392; y = 1386; z = -1056,$$

qui denuo, reiectis signis, per 2 deprimuntur, ita ut

$$x = 196; y = 693; z = 528.$$

Supra autem iam multo minores numeros nacti sumus.

Operat.II. §16. Ut praeter primum terminum etiam duo ultimi tollantur statuamus:

$$R = 2mpp + 2npq + (1 - nn)qq;$$

unde orietur sequens aequatio:

$$\frac{p}{q} = \frac{4m - 4mnn - m^4}{4mn(2+m)}, \text{ sive}$$

$$\frac{p}{q} = \frac{8 - 4mm - m^3}{4n(2+m)}, \text{ ob } nn = mm - 1,$$

sive etiam

$$\frac{p}{q} = \frac{(2+m)(4 - 2m - mm)}{4n(2+m)} = \frac{4 - 2m - mm}{4n}.$$

Fiat ut supra

$$p = \Delta(4 - 2m - mm) \text{ et } q = 4\Delta n$$

hincque erit

$$r = p - nq = \Delta(8 - 2m - 5mm);$$

(xvii) Final) inserted by translator.

denique

$$x = pp + qq - rr; y = 2pr; z = 2qr;$$

$$P = pp + qq + rr \text{ et } Q = 2qrR,$$

existente

$$R = 2mpp + 2npq + (1 - nn)qq.$$

§17. Sumamus iterum, quo res exemplo illustretur, $n = \frac{3}{4}$, ideoque $m = \pm\frac{5}{4}$, fietque,

$$p = \Delta(\tfrac{39}{16} \mp \tfrac{5}{2}) \text{ et } q = 3\Delta.$$

Sumatur $\Delta = 16$, et signo superiore valente erit $p = -1$ et $q = 48$; hinc fit $r = -37$, unde numeri quaesiti prodeunt

$$x = 936; y = 74; z = 3552,$$

sive deprimendo

$$x = 468; y = 37; z = 1776,$$

qui praecedentibus adhuc maiores sunt.

§18. Tollamus nunc tres terminos priores, ponendo Operat. III.

$$R = 2mpp - mnpq + \tfrac{m^4+3nn}{4m}qq,$$

ex quo haec resultat aequatio:

$$g\left(\tfrac{(m^4+3nn)^2}{16mm} - (1-nn)^2\right) = -\tfrac{n}{2}(m^4 - 5nn + 8)p.$$

Ex hac autem forma iam satis manifestum est, nullos numeros minores, Problemati satisfacientes, elici posse; quamobrem ulteriore evolutione supersedemus.