Excerpt of a Letter from Euler to Beguelin[1]

May 1778

I heard with pleasure the lecture on the thesis of M. Beguelin, concerning prime prime numbers, which was inserted into the latest Volume of Theses of the Royal Academy of Berlin. As I have worked for some time on the same subject, I think that he will receive with much satisfaction some observations that I have had the occasion to make relative to the problem that he dealt with in the aforementioned paper.

His research is founded on the beautiful property that all numbers which can only be represented as x^2+y^2 in one way are either prime or twice a prime, taking the numbers x and y to be relatively prime. I have already remarked that several other similar formulas of the form nx^2+y^2 are endowed with the same property, and that, provided that one gives the letter n a *convenient* value, for example, $2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13$ etc., one always derives a prime number; or rather, that by excluding the following values of n: $11, 14, 17, 19, 20, 23, 26, 27$ etc., the formula nx^2+y^2 always yields prime numbers; because the number 15, for example, which is expressible only one way by the formula $11x^2+y^2$, is composite. It is thus of the numbers which I have just excluded and not of those which I have named convenient values and surely giving a prime, all of which are expressible in only one way as nx^2+y^2. It is therefore of the last importance to distinguish between those values of n which are convenient and those which must be excluded in this research.

To this end I have found and demonstrated this rule: that if all numbers which can be represented as $n+y^2$ and are less than $4n$ (taking y to be relatively prime to n) are either a prime p, twice a prime $2p$, the square of a prime p^2, or finally, some power of 2, then the value of n, which satisfies these conditions, is able to be admitted as convenient. For example, I found that the number 60 lies in the series of convenient values. This is because $60+1^2=61=p$, $60+7^2=109=p$, $60+11^2=181=p$ and $60+13^2=229=p$. It is necessary to stop at this point because the following values surpass the limit 4×60. It is the same with the number 15, as $15+1^2=16=2^4$ and $15+4^2=31=p$.

By using this rule I have been able to find with ease all values which one can give the letter n, in order that every number which can be represented as nx^2+y^2 in only one way can be taken to be prime. Here are the values:

[1]Translation from original French by Benjamin Linowitz, Dartmouth College

1	16	48	120	312
2	18	57	130	330
3	21	58	133	345
4	22	60	165	357
5	24	70	168	385
6	25	72	177	408
7	28	78	190	462
8	30	85	210	520
9	33	88	232	760
10	37	93	240	840
12	40	102	253	1320
13	42	105	273	1365
15	44	112	280	1848.

These numbers, which, far from being selected at random, follow a law of progression, which is evident enough when we look through all the successive exclusions which must be passed in order to find convenient values, seem to go to infinity; I was therefore very surprised to see them stop at 1848, beyond which I have only found incompatible values. However, by using the final value 1848, one is able to discover extremely large prime numbers, as nothing can be easier to check than whether or not a proposed number is represented as $1848x^2 + y^2$ in only one way, and in the first case one can heartily declare the number to be prime. By means of this form I have found the following numbers to be prime: 1016401, 1103257, 1288057, 1487641, 1702009, 2995609, 4658809, 9094009, 11866009 and 18518809. In the other case, when the proposed number is able to be represented as $1848x^2 + y^2$ in more than one way, it would be superfluous to remark that one can easily determine the divisors of the number. I would like to add that an error has found its way into the table of prime numbers in Volume XIX of the Commentaries of our Academy, stemming from having neglected the divisor 293, which has an affect on the number 100009, which must be erased from the list, being equal to 293×3413.