On a Geographic Projection of the Surface of the Sphere[*]

Leonhard Euler

1. In the previous work I have derived all possible methods of taking an image of a spherical area in a plane, so that the smallest parts are reproduced through similar figures. From this followed immediately the construction of Mercator's sea chart, as well as the maps of the northern and southern hemisphere[1]But how today's usual construction of hemispheres[2], which appear as upper and lower from an arbitrary point, follows from my formulas, was not completely evident, although these maps too possess the above-mentioned property. This has caused me to inquire more exactly how the last-named method of representation is connected with the general formulas set forth there, and how best it can be derived from them.

2. The general formulae, which I have for that kind of map sketch have developed, are the following[3]. For any point on the sphere, let v be the distance from the pole, t be the distance from a chosen meridian of origin along the same latitude, and let x and y be the rectilinear coordinates which

[*]Translation of Euler's "de Proiectione Geographica Superficiei Sphaericae". Translation date May 2008. (*Opera Omnia*, ser. 1, vol. 28, pp. 248–275) Translator's name and email address: George W. Heine, <gheine@mathnmaps.com>

[1]Euler refers to the polar stereographic projection, in which the origin is the geographic north or south pole.

[2]The stereographic projection in which the origin is a point other than a pole. Placing the origin on the equator was a commonly used projection for world maps in the XVIIIth century. An example can be viewed in the Euler Archives at <http://www.math.dartmouth.edu/ euler/atlas/map02.jpg>

[3]These are developed in Paragraph 44 of "De repraesentatione superficiei sphaericae super plano"[E 490]

the position of the corresponding point on the plane determines, so that

$$x = \Delta\left[\log\cot(\tfrac{1}{2}v) + t\sqrt{-1}\right] + \Delta\left[\log\cot(\tfrac{1}{2}v) - t\sqrt{-1}\right], \tag{2.1}$$

$$y\sqrt{-1} = \Delta\left[\log\cot(\tfrac{1}{2}v) + t\sqrt{-1}\right] - \Delta\left[\log\cot(\tfrac{1}{2}v) - t\sqrt{-1}\right]. \tag{2.2}$$

One can rewrite the first of these equations in the following manner [4]:

$$x = \Delta\left[\cot(\tfrac{1}{2}v)(\cos t + \sqrt{-1}\sin t)\right] + \Delta\left[\cot(\tfrac{1}{2}v)(\cos t - \sqrt{-1}\sin t)\right] \tag{2.3}$$

and similarly with the second. Moreover one observes that

$$\frac{1}{\cot(\tfrac{1}{2}v)(\cos t \pm \sqrt{-1}\sin t)} = \tan(\tfrac{1}{2}v)(\cos t \mp \sqrt{-1}\sin t), \tag{2.4}$$

so that the previous formulae can be given in the following form[5]:

$$x = \Delta\left[\tan(\tfrac{1}{2}v)(\cot t + \sqrt{-1}\sin t)\right] + \Delta\left[\tan(\tfrac{1}{2}v)(\cot t - \sqrt{-1}\sin t)\right], \tag{2.5}$$

$$y\sqrt{-1} = \Delta\left[\tan(\tfrac{1}{2}v)(\cot t + \sqrt{-1}\sin t)\right] - \Delta\left[\tan(\tfrac{1}{2}v)(\cot t - \sqrt{-1}\sin t)\right]. \tag{2.6}$$

We allow the sign Δ, which denotes an indeterminate function, to change between these representations. The first pair of equations yield the formulae for Sea Charts[6], while the latter two yield the formulae for the maps of the northern and southern hemisphere[7].

3. Now in order to more easily establish, how also the above projections, which are based on on the same principle, can be derived out of our formulae, I wish to fully develop the main features of the projection, which one customarily takes care to designate as stereographic. With this projection the spherical surface is projected on to a tangent plane, as it appears, according to the rules of perspective, to an observer located at the point on the sphere

[4] In (2.3), Euler uses the symbol Δ to represent the composition of the function Δ in (2.1) and (2.2) with the complex natural logarithm function, implicitly assuming the argument is nonzero.

[5] In (2.5) and (2.6), Euler uses the symbol Δ to represent the composition of the function Δ in (2.3) with the multiplicative reciprocal function, implicitly assuming the argument is nonzero

[6] Mercator's projection

[7] Polar aspect of the stereographic projection

opposite the point of tangency.[8] Let the circle AMC of the sphere and the line EF of the plane which the circle touches at C be represented. Then the location of the observer is the point A, opposite to point C. Now on the sphere we take arbitrarily the point M, and extend the straight line AMS, which connects A with M, to meet the line EF in the point S, then S is the projection of M. Furthermore, we set the radius of the circle =1, so that the diameter AC=2, and designate the arc CM by z, so that the angle $CAM = \frac{1}{2}z$, and the distance

$$CS = 2\tan(\tfrac{1}{2}z) = \frac{2\sin z}{1+\cos z} + 2\sqrt{\frac{1-\cos z}{1+\cos z}}. \tag{3.1}$$

4. From M to AC drop the perpendicular MP so that MP=$\sin z$ (Fig. 1). Now one lets the plane figure rotate about the axis AC, so that M describes a circle, whose plane is parallel to the tangent plane and whose radius is $= \sin z$; to this circle corresponds, in the tangent plane, a described circle with radius $CS = 2\tan(\frac{1}{2}z)$. The radius of the circle on the sphere is thus to the radius of its projection as PM to CS, or as AP to AC, or finally as AM to AS. Furthermore, a central angle in the described circle of radius PM on the sphere is equal to the central angle of its projection on the plane.

5. Now we consider a point m on the sphere very near the point M, whose projection is s, so that the small arc Mm corresponds to the small segment Ss. Then we ask, how the elements Mm and Ss are related. To this end we next observe that the angle $ASC = 90° - \frac{1}{2}z = AsC$. Furthermore, the measure of the angle AMm is half of the arc AM; that is, $AMm = 90° - \frac{1}{2}z$ and therefore equal to the angle AsC. It follows that the triangles AMm and AsS are similar, and therefore

$$Mm : Ss = AM : AS, \qquad \text{that is,} \qquad = AP : AC.$$

This proportion is the same as that which we found between the radius PM on the circle described on the sphere and the radius CS of the corresponding circle on the plane. Thus the arc elements are related as the radii of these

[8] The projective plane can also be taken as any plane, not containing the observation point, which is parallel to the tangent plane described by Euler. A common choice, used by Ptolemy and others, was to take the equatorial plane which passes through the great circle with pole at the observation point. The only effect is to scale everything by a constant.

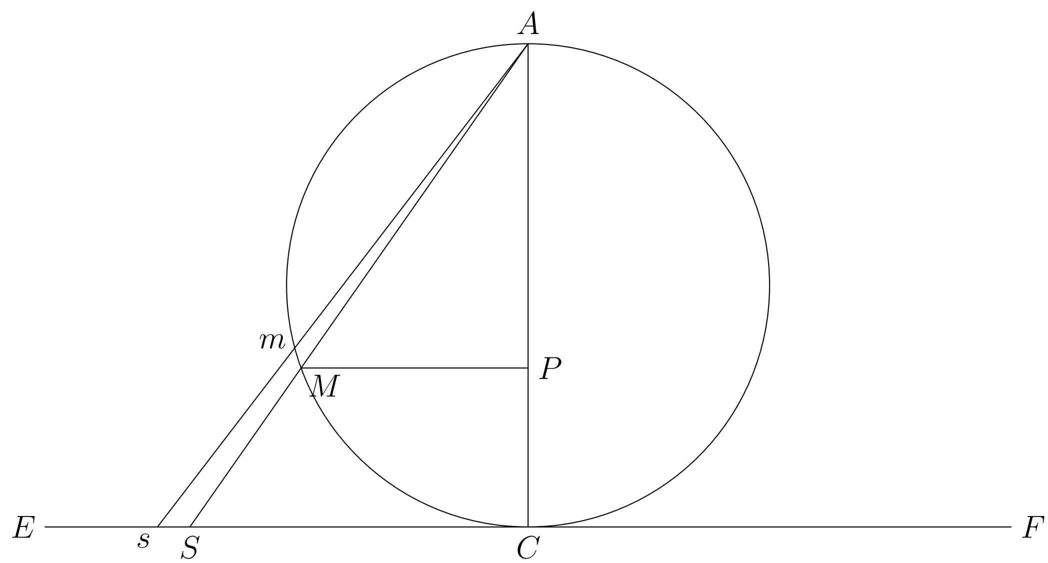

Figure 1: A copy of Euler's original Figure 1

circles. From this it follows, if we conceive of Mm as an infinitely small piece of the spherical surface, the projection of the same observed piece is similar. The projection follows with it the same rule from which I have derived my general formulae.

6. As before, let the circle AGC (Fig. 2) represent the sphere, whose surface is to be projected on to the plane tangent at C. Let a pole of the earth lie at the point G. The point H on the plane corresponds to this pole. The distance H from C is

$$CH = 2\tan(\tfrac{1}{2}g), \tag{6.1}$$

where g is the arc CG. An arbitrary point M on the sphere is separated from the pole be the distance $GM = v$, while the angle $CGM = t$ is the longitude of the point M, relative to the meridian GC considered as the meridian of origin. Finally we consider the great circle containing CM. Now S is that point of the projection which corresponds to the point M, so that $CS = 2\tan(\frac{1}{2}CM)$ and the angle ECS equals the angle GCM. To determine the position of the point S must one calculate the side CM and the angle GCM of the spherical triangle GCM.

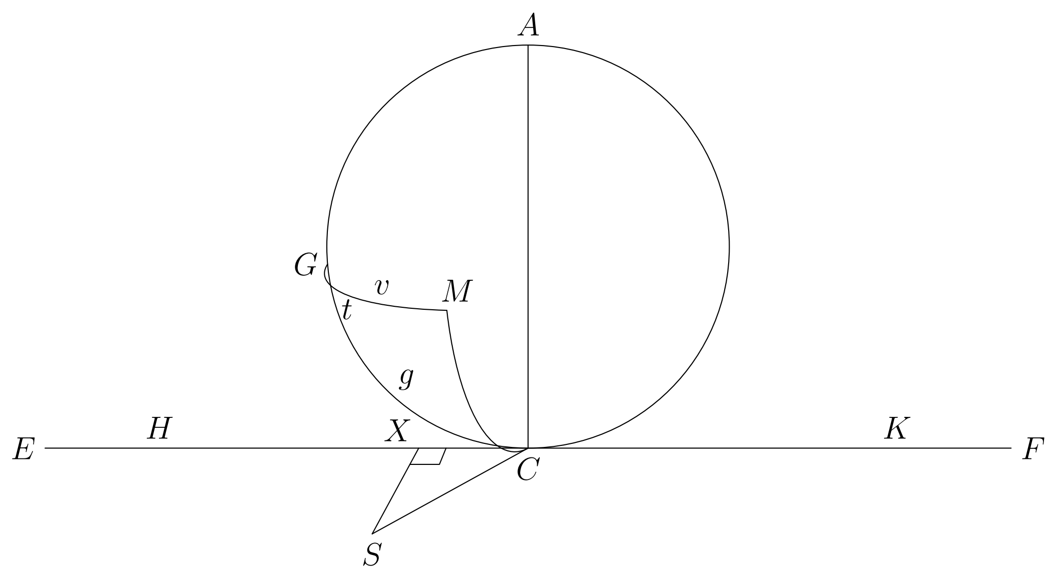

Figure 2: A modified version of Euler's original Figure 2. Modifications include: shifting the point S to the left, to better render the drawing in perspective, and adding the labels v (the arc GM), t (the spherical angle CGM), and g (the arc GM).

7. In the spherical triangle CGM there are known two sides, $CG = g$ and $GM = v$ and their enclosed angle, $MGC = t$. The basic formula of spherical trigonometry therefore yields

$$\cos CM = \cos g \cos v + \sin g \sin v \cos t, \tag{7.1}$$

and since[9]

$$CS = 2\tan(\tfrac{1}{2}CM) = \frac{2\sin CM}{1+\cos CM} = 2\sqrt{\frac{1-\cos CM}{1+\cos CM}} \tag{7.2}$$

we obtain

$$CS = 2\sqrt{\frac{1-\cos g\cos v - \sin g \sin v \cos t}{1+\cos g\cos v + \sin g \sin v \cos t}}. \tag{7.3}$$

Furthermore the equation

$$\tan GCM = \frac{\sin v \sin t}{\cos v \sin g - \sin v \cos g \cos t} \tag{7.4}$$

yields at the same time the angle ECS of the projection.

8. Now, from the point S of the projection, we drop the perpendicular SX onto the baseline EF, wherein lies the pole H, and denote the coordinates CX and SX by x and y, respectively. Then since

$$CS = \frac{2\sin CM}{1+\cos CM}, \tag{8.1}$$

we have that

$$x = \frac{2\sin CM \cdot \cos GCM}{1+\cos CM}, \quad y = \frac{2\sin CM \cdot \sin GCM}{1+\cos CM}, \tag{8.2}$$

and from this it follows that

$$\frac{x}{y} = \tan GCM = \frac{\sin v \sin t}{\cos v \sin g - \sin v \cos g \cos t}. \tag{8.3}$$

Moreover it follows from the equations found above[10] that:

$$x^2 + y^2 = CS^2 = \frac{4(1-\cos v\cos g - \sin v \sin g \cos t)}{1+\cos v \cos g + \sin v \sin g \cos t}. \tag{8.4}$$

With this one has two different expressions for calculating the coordinates x and y.

[9] (3.1)
[10] (7.3)

9. We can find the value of these coordinates even more easily in the following way. From the equality

$$\sin t : \sin CM = \sin GCM : \sin v \tag{9.1}$$

it follows that

$$\sin CM \cdot \sin GCM = \sin v \cdot \sin t. \tag{9.2}$$

Using this equation together with the previously introduced value[11]

$$\tan GCM = \frac{\sin GCM}{\cos GCM} = \frac{\sin CM \cdot \sin GCM}{\cos v \sin g - \sin v \cos g \cos t}, \tag{9.3}$$

whence

$$\sin CM \cdot \cos GCM = \cos v \sin g - \sin v \cos g \cos t, \tag{9.4}$$

from which we obtain[12] the values

$$x = \frac{2(\cos v \sin g - \sin v \cos g \cos t)}{1 + \cos CM}, \qquad y = \frac{2 \sin v \sin t}{1 + \cos CM}. \tag{9.5}$$

Finally we substitute the value[13],

$$\cos CM = \cos g \cos v + \sin g \sin v \cos t, \tag{9.6}$$

so that we obtain the following expressions for the coordinates:

$$x = \frac{2(\cos v \sin g - \sin v \cos g \cos t)}{1 + \cos g \cos v + \sin g \sin v \cos t}, \tag{9.7}$$

$$y = \frac{2 \sin v \sin t}{1 + \cos g \cos v + \sin g \sin v \cos t}. \tag{9.8}$$

10. Setting $v = 0$ in these formulae, one gets the the coordinates of the point which the pole H of the projection takes on. For this,

$$x = \frac{2 \sin g}{1 + \cos g} = 2 \tan(\tfrac{1}{2} g) = CH, \qquad y = 0. \tag{10.1}$$

[11] (7.4)
[12] using (8.2)
[13] (7.1)

Also the place of the other pole can easily be indicated; it is only necessary to set $v = 180°$. For this case, one gets

$$x = \frac{-2\sin g}{1-\cos g}, \qquad y = 0. \tag{10.2}$$

Let K be this second pole; then

$$CK = \frac{2\sin g}{1-\cos g} = 2\cot(\tfrac{1}{2}g). \tag{10.3}$$

Furthermore, taking $CE = CF = 2$, EF becomes the diameter of the circle inside of which the entire half-sphere centered about C is depicted. The diameter of this circle is 4, i.e., twice as large as the diameter of the sphere.

11. In order to find the Equator in our Projection, we take $v = 90°$; then x and y represent a point on the equator of the map, and[14]

$$x = \frac{-2\cos g\cos t}{1+\sin g\cos t}, \qquad y = \frac{2\sin t}{1+\sin g\cos t}. \tag{11.1}$$

From the formula established above[15],

$$x^2 + y^2 = \frac{4(1-\sin g\cos t)}{1+\sin g\cos t}, \tag{11.2}$$

and therefore

$$\frac{x}{x^2+y^2} = \frac{-\cos g\cos t}{2(1-\sin g\cos t)}, \tag{11.3}$$

thus

$$\cos t = \frac{2x}{2x\sin g - (x^2+y^2)\cos g}; \tag{11.4}$$

setting this value in the equation for x[16], one gets

$$4x\sin g - (x^2+y^2)\cos g = -4\cos g. \tag{11.5}$$

Thus we have

$$x^2 + y^2 = \frac{4(x\sin g + \cos g)}{\cos g} \tag{11.6}$$

[14] From (9.6) and (9.7)
[15] (8.4)
[16] first equality in (11.1)

and also

$$y^2 + (2\tan g - x)^2 = \frac{4}{\cos^2 g}. \tag{11.7}$$

From this one sees that on the map the equator becomes a circle of radius $\dfrac{2}{\cos g}$. In order to find the center of this circle, one marks off (Fig. 3) the distance $CJ = 2\tan g$ on the x axis, whereby $JX = 2\tan g - x$, so that[17]

$$XS^2 + JX^2 = \frac{4}{\cos^2 g}. \tag{11.8}$$

It follows that $JS = 2/\cos g$; thus the length JS is constant[18]. The point J becomes the center of the circle corresponding to the equator,so that $CJ = 2\tan g$. Now erect at C the perpendicular $CD = 2$, and that[19] $JD = 2/\cos g$. Thus one obtains the equator on the map by describing a circle around J of radius JD.

12. Now we wish to determine the Circles of Parallel on our map. In order to avoid some tedium in the calculation, the following abbreviations are given:

$$\begin{aligned} a &= 2\sin g\cos\alpha, & b &= 2\cos g\sin\alpha, \\ c &= 1 + \cos g\cos\alpha, & d &= \sin g\sin\alpha, \\ e &= 4 - 4\cos g\cos\alpha. \end{aligned}$$

Here we use the letter α in place of the earlier letter v, so that α denotes the distance from the pole of the Parallel Circle under consideration. Then our equations[20] take the form

$$x = \frac{a - b\cos t}{c + d\cos t}, \qquad x^2 + y^2 = \frac{e - 4d\cos t}{c + d\cos t}. \tag{12.1}$$

From the first follows

$$\cos t = \frac{a - cx}{b + dx}, \tag{12.2}$$

[17] As noted at the beginning of par. 8, y represents the distance XS.
[18] not dependent on the longitude t
[19] the angle $JDC = g$ and therefore
[20] (9.7) and (8.4)

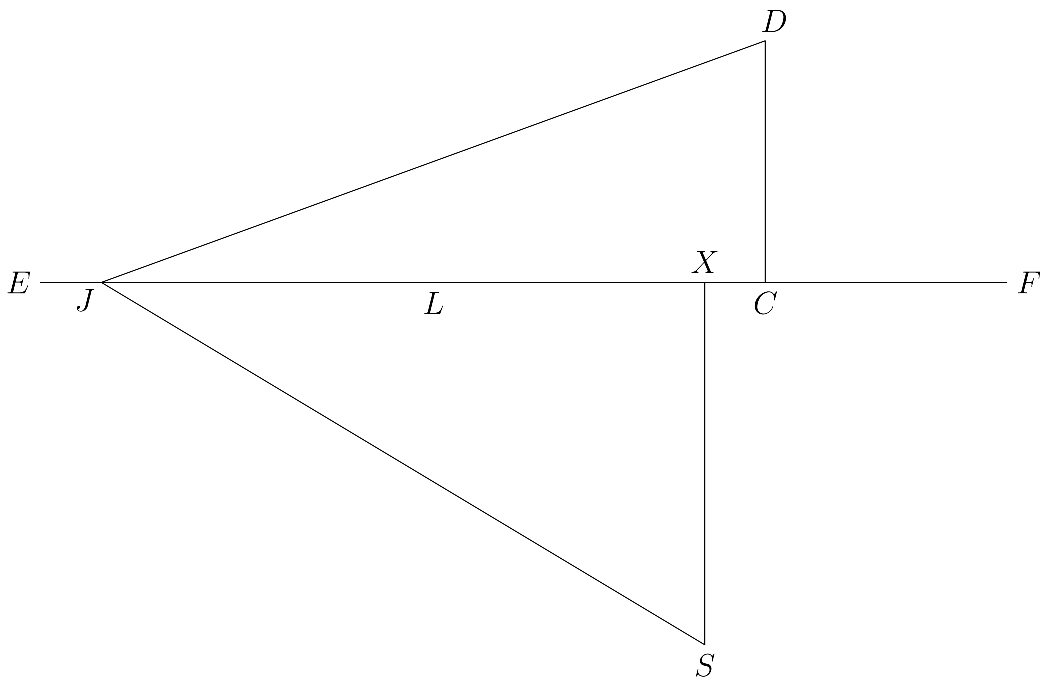

Figure 3: A copy of Euler's original Figure 3

and substituting this into the second equation,

$$x^2 + y^2 = \frac{d(e+4c)x + be - 4ad}{bc + ad} \tag{12.3}$$

Expressing a, b, c, d again in terms of g and α, one obtains

$$x^2 + y^2 = \frac{4[x \sin g + \cos g - \cos\alpha]}{\cos g + \cos\alpha}. \tag{12.4}$$

Bringing this equation into the form

$$y^2 + \left(\frac{2\sin g}{\cos g + \cos\alpha} - x\right)^2 = \frac{4\sin^2\alpha}{(\cos g + \cos\alpha)^2}, \tag{12.5}$$

and from this one recognizes that the Parallel Circle under consideration is a circle of radius $\dfrac{2\sin\alpha}{\cos g + \cos\alpha}$, with center on the axis EF at the point L, and whose distance from the point C is

$$CL = \frac{2\sin g}{\cos g + \cos\alpha}. \tag{12.6}$$

13. Now we wish to investigate the Projections of all Meridians (Fig. 2). In the first place, $t = 0$ whenever $y = 0$; that is, the straight line HK represents the principal Meridian, from which the other longitudes are counted. Furthermore, let β be the inclination of the desired Meridian with respect to the principal Meridian, so that $t = \beta$ and our equations [21] become

$$x = \frac{2(\sin g \cos v - \cos\beta \cos g \sin v)}{1 + \cos g \cos v + \cos\beta \sin g \sin v}, \tag{13.1}$$

$$y = \frac{2\sin\beta \sin v}{1 + \cos g \cos v + \cos\beta \sin g \sin v}, \tag{13.2}$$

$$x^2 + y^2 = \frac{4(1 - \cos g \cos v - \cos\beta \sin g \sin v)}{1 + \cos g \cos v + \cos\beta \sin g \sin v}; \tag{13.3}$$

and from these equations the quantity v is to be eliminated. To this end, we divide the first two, so that

$$\frac{y}{x} = \frac{\sin\beta \sin v}{\sin g \cos v - \cos\beta \cos g \sin v} = \frac{\sin\beta \tan v}{\sin g - \cos\beta \cos g \tan v}, \tag{13.4}$$

[21] (9.6),(9.7), and (8.4)

and from this it follows that

$$\tan v = \frac{y \sin g}{x \sin \beta + y \cos \beta \cos g}. \tag{13.5}$$

14. Now in order to most easily use this value in the remaining equations, we construct the following:

$$4 - x^2 - y^2 = \frac{8 \cos g \cos v + 8 \cos \beta \sin g \sin v}{1 + \cos g \cos v + \cos \beta \sin g \sin v}; \tag{14.1}$$

dividing through by y, we obtain

$$\frac{4 - x^2 - y^2}{y} = \frac{4 \cos g \cos v + 4 \cos \beta \sin g \sin v}{\sin \beta \sin v} \tag{14.2}$$

$$= \frac{4 \cos g + 4 \cos \beta \sin g \tan v}{\sin \beta \tan v} \tag{14.3}$$

Here we replace $\tan v$ by the value obtained above [22], to obtain

$$\frac{4 - x^2 - y^2}{y} = \frac{4y \cos \beta + 4x \sin \beta \cos g}{y \sin \beta \sin g}, \tag{14.4}$$

and from this it follows that

$$x^2 + y^2 = 4 - \frac{4y \cos \beta + 4x \sin \beta \cos g}{\sin \beta \sin g}, \tag{14.5}$$

which is the equation of a circle. With this one can conclude in the same manner that all great circles on the sphere are represented as circular arcs, or straight lines, on the map.

15. Now in order to ascertain (Fig 4) the center as well as the radius of each Meridian assigned by our projection, we recast the equation in the following form:

$$\left(\frac{2 \cos g}{\sin g} + x\right)^2 + \left(\frac{2 \cos \beta}{\sin \beta \sin g} + y\right)^2 = \frac{4}{\sin^2 \beta \sin^2 g}. \tag{15.1}$$

[22] (13.5)

If therefore, the points H and K are poles on the map, then

$$CH = 2\tan(\tfrac{1}{2}g) = \frac{2\sin g}{1+\cos g}, \tag{15.2}$$

$$CK = 2\cot(\tfrac{1}{2}g) = \frac{2\sin g}{1-\cos g}, \tag{15.3}$$

so that the whole distance is

$$HK = \frac{4}{\sin g}, \qquad \tfrac{1}{2}HK = \frac{2}{\sin g}, \tag{15.4}$$

and if O is the midpoint of HK, then

$$CO = \frac{2\cos g}{\sin g}; \tag{15.5}$$

furthermore, since CX was designated as x,

$$OX = \frac{2\cos g}{\sin g} + x. \tag{15.6}$$

From the point O on the axis, the perpendicular

$$ON = \frac{2\cos\beta}{\sin\beta\sin g}, \tag{15.7}$$

and setting $XL = ON$, we have

$$SL = \frac{2\cos\beta}{\sin\beta\sin g} + y. \tag{15.8}$$

Therefore,

$$OX^2 + LS^2 = LN^2 + SL^2 = NS^2 = \frac{4}{\sin^2\beta}\sin^2 g, \tag{15.9}$$

that is,

$$NS = \frac{2}{\sin\beta\sin g}. \tag{15.10}$$

Now, since this radius equals exactly NH, One recognizes from this, that the point N is the center of the Meridian on the map, its radius is 2 $\sin\beta\sin g$, and NH has exactly the same length. Since the Meridian was arbitrary, we have shown that the representations of all meridian circle pass through the two poles.

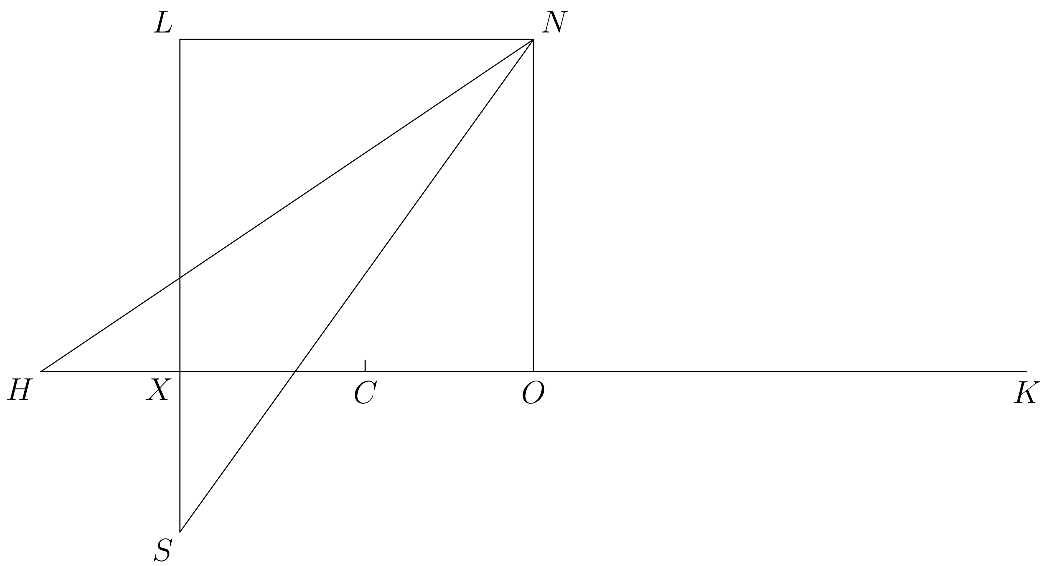

Figure 4: A copy of Euler's original Figure 4

16. The question is now asked, which form one must give the function Δ (§2) in order that the Projection under consideration be obtained. First of all, one recognizes that higher powers than the first (of the arguments) can not occur; otherwise, multiple values of the angles t and v would appear. Therefore the said function must be a fraction, that yields, as above[23], fractions for the expressions of x and y. Therefore we want $\Delta(z)$ to take the following general form:

$$\Delta(z) = \frac{a + bz}{c + dz}, \tag{16.1}$$

while we choose for z the last of the above indicated forms[24], namely

$$z = \tan(\frac{1}{2}v) \cdot (\cos t \pm \sqrt{-1}\sin t). \tag{16.2}$$

Accordingly, we consider the function

$$\frac{a + b\tan(\frac{1}{2}v) \cdot (\cos t \pm \sqrt{-1}\sin t)}{c + d\tan(\frac{1}{2}v) \cdot (\cos t \pm \sqrt{-1}\sin t)} \tag{16.3}$$

and replace in it $\tan(\frac{1}{2}v)$ by $\sin v/(1 + \cos v)$, so that it takes the following form:

$$\frac{a(1 + \cos v) + b\sin v \cdot (\cos t \pm \sqrt{-1}\sin t)}{c(1 + \cos v) + d\sin v \cdot (\cos t \pm \sqrt{-1}\sin t)}. \tag{16.4}$$

17. In order to fashion the calculation more clearly, we write the preceding fraction more simply as

$$\frac{P \pm Q\sqrt{-1}}{R \pm S\sqrt{-1}},$$

where

$$P = a(1 + \cos v) + b\sin v\cos t, \qquad Q = b\sin v\sin t, \tag{17.1}$$
$$R = c(1 + \cos v) + d\sin v\cos t, \qquad S = d\sin v\sin t. \tag{17.2}$$

[23] (9.6) and (9.7)
[24] (2.5) and (2.6)

Then, for the coordinates x, y we have the following expressions:

$$x = \frac{P + Q\sqrt{-1}}{R + S\sqrt{-1}} + \frac{P - Q\sqrt{-1}}{R - S\sqrt{-1}}, \tag{17.3}$$

$$y\sqrt{-1} = \frac{P + Q\sqrt{-1}}{R + S\sqrt{-1}} - \frac{P - Q\sqrt{-1}}{R - S\sqrt{-1}}. \tag{17.4}$$

This yields

$$x = \frac{2PR + 2QS}{R^2 + S^2}, \qquad y = \frac{2QR - 2PS}{R^2 + S^2}. \tag{17.5}$$

18. Now we replace again P, Q, R, S with their values, and obtain for the common denominator:

$$R^2 + S^2 = c^2(1 + \cos v)^2 - 2cd(1 + \cos v)\sin v \cos t + d^2 \sin^2 v \tag{18.1}$$

$$= (1 + \cos v)[c^2(1 + \cos v) + 2cd \sin v \cos t + d^2(1 - \cos v)]. \tag{18.2}$$

The factors in the numerators of x and y become

$$PR + QS = (1 + \cos v)[ac(1 + \cos v) + (bc + ad)\sin v \cos t + bd(1 - \cos v)], \tag{18.3}$$

$$QR - PS = (1 + \cos v)(bc - ad)\sin v \sin t. \tag{18.4}$$

With this we obtain the following expressions for the coordinates:

$$x = \frac{2ac(1 + \cos v) + 2(bc + ad)\sin v \cos t + 2bd(1 - \cos v)}{c^2(1 + \cos v) + 2cd \sin v \cos t + d^2(1 - \cos v)}, \tag{18.5}$$

$$y = \frac{2(bc - ad)\sin v \sin t}{c^2(1 + \cos v) + 2cd \sin v \cos t + d^2(1 - \cos v)}. \tag{18.6}$$

19. We compare these formulae[25] with those which we found above[26], that is

$$x = \frac{2(\cos v \sin g - \sin v \cos g \cos t)}{1 + \cos g \cos v + \sin g \sin v \cos t},$$

$$y = \frac{2 \sin v \sin t}{1 + \cos g \cos v + \sin g \sin v \cos t},$$

[25] (18.5) and (18.6)

[26] (9.7) and (9.8)

and thus we see, that the latter forms agree with the former, and one can now easily discover the values which one must join to the constants a, b, c, d, in order to complete the agreement. In order that the denominators be identical, we must have

$$c^2 + d^2 = 1, \qquad c^2 - d^2 = \cos g, \qquad 2cd = \sin g. \tag{19.1}$$

From the first two of these equations are obtained

$$c^2 = \frac{1 + \cos g}{2} = cos^2(\tfrac{1}{2}g), \qquad d^2 = \frac{1 - \cos g}{2} = sin^2(\tfrac{1}{2}g); \tag{19.2}$$

that is,

$$c = cos(\tfrac{1}{2}g), \qquad d = \sin(\tfrac{1}{2}g), \tag{19.3}$$

and the third equation is automatically fulfilled:

$$2cd = 2\sin(\tfrac{1}{2}g)\cos(\tfrac{1}{2}g) = \sin g. \tag{19.4}$$

In order that the numerators in the two expressions for x be identical, it is necessary that

$$ac + bd = 0, \qquad ac - bd = \sin g, \qquad bc + ad = -\cos g, \tag{19.5}$$

or, if one substitutes in the above values for c and d:

$$a\cos(\tfrac{1}{2}g) + b\sin(\tfrac{1}{2}g) = 0, \tag{19.6}$$

$$a\cos(\tfrac{1}{2}g) - b\sin\tfrac{1}{2}g = \sin g, \tag{19.7}$$

$$b\cos(\tfrac{1}{2}g) + a\sin(\tfrac{1}{2}g) = -\cos g. \tag{19.8}$$

The first two equations yield

$$a = \frac{\sin g}{2\cos(\tfrac{1}{2}g)} = \sin(\tfrac{1}{2}g), \tag{19.9}$$

$$b = \frac{-\sin g}{2\sin(\tfrac{1}{2}g)} = -\cos(\tfrac{1}{2}g), \tag{19.10}$$

and these two values suffice to satisfy the third equality. It remains only to examine whether the values we have found also are able to satisfy the two expressions for the values of y. For this it is necessary that

$$bc - ad = 1. \tag{19.11}$$

But with the values we have found, $bc = -\cos^2(\frac{1}{2}g)$ and $ad = \sin^2(\frac{1}{2}g)$, so that

$$bc - ad = -1. \tag{19.12}$$

However, it is observed that one can exchange the positive and negative coordinate axes, so that the agreement is complete.

20. From the foregoing discussion one perceives that our general formulae lead to the stereographic projection, if the function $\Delta(z)$ takes the form

$$\Delta(z) = \frac{\sin(\frac{1}{2}g) - z\cos(\frac{1}{2}g)}{\cos(\frac{1}{2}g) + z\sin(\frac{1}{2}g)} = \frac{\tan(\frac{1}{2}g) - z}{1 + z\tan(\frac{1}{2}g} \tag{20.1}$$

Moreover, let it be remarked, that this method of projection is extraordinarily appropriate for the practical applications required by Geography, for it does not strongly distort any region of the earth. It is also important to note that with this projection, not only are all Meridians and Circles of Parallel exhibited as circles or as straight lines, but all great circles on the sphere are expressed as circular arcs or straight lines. Other hypotheses, which one might perhaps make concerning the function Δ, will not possess this straightforward advantage.

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