On the mapping of Spherical Surfaces onto the Plane

Leonhard Euler

1777

TRANSLATOR'S NOTE

In translating this work, I am deeply indebted to the work of A. Wangerin, whose thorough and scholarly 1897 German translation appears in [Wan1897]. However, I believe that Wangerin was writing largely for an audience of mathematicians, and that his goal was to explain Euler's reasoning in the "modern" terms of his era. Thus, for example, Wangerin uses the term "total differential", even though (as Wangerin notes) this wording was not yet used in Euler's time. Today, in contrast, the interested audience likely consists of both mathematicians and historians of mathematics. To satisfy the latter, I have attempted, as near as possible, to mimic Euler's phrasing, and especially his mathematical notation. I have made a few exceptions in cases where Euler's notation might be confusing to a modern reader. These include:

- Use of "ln()" rather than "l", to denote the natural or "hyperbolic" logarithm function;
- Use of "arcsin, arccos" rather than "A sin, A cos" for the inverse trigonometric functions (particularly confusing since Euler uses a constant A in his computations);
- Use of, for example, "$\cos^2 u$" rather than "$\cos u^2$" to denote the square of the cosine;
- Introduction of parentheses to delimit the argument of a function where this was not clear from the context.

In most other cases, I have attempted to copy Euler's original notation—in particular, preserving his use of the differential operator d, and the symbol $\sqrt{-1}$, since these had a slightly different meaning to Euler than they would have to mathematical practicioners today.

George Heine

gheine@mathnmaps.com

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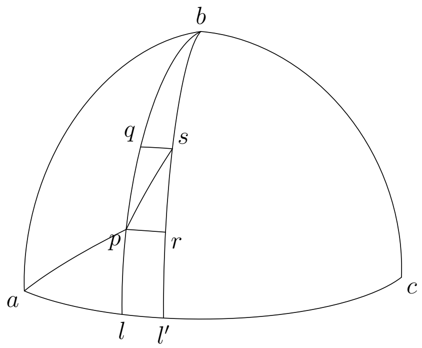

Figure 1

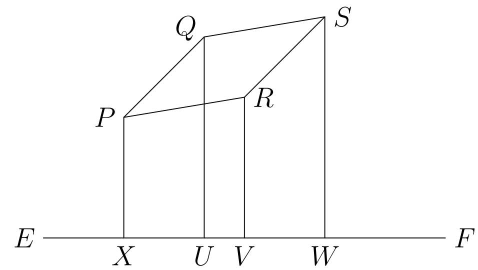

Figure 2

1. In the following I consider not only optical projections, in which the different points of a sphere are represented on a plane as they appear to an observer at a specified place; in other words, representations by which a single point visible to the observer is projected onto the plane by the rules of perspective; indeed, I take the word "mapping" in the widest possible sense; any point of the spherical surface is represented on the plane by any desired rule, so that every point of the sphere corresponds to a specified point in the plane, and inversely; if there is no such correpsondence, then the image of a point from the sphere is imaginary.

2. In Figure 1, abc represents a portion of the Spherical surface, whose pole is at the point b and Equator the circle alc. Let ab be the prime Meridian, from which, as is usual in Geography, the longitude of a point on the Sphere is measured. Now consider an arbitrary point p, which lies on the Meridian bpl; this Meridian forms an angle abl with the prime Meridian; it is equal to the Equatorial arc $al = t$. The latitude of the point is the arc $lp = u$, if one takes the radius of the Sphere to be unity.

In Figure 2 the plane of the figure is that in which the image lies, and P is the point which corresponds to p on the Sphere. From P drop the perpendicular PX to the axis of abscissa EF, whose position may be freely chosen, with the point E as origin, and take the abscissa $EX = x$ and the ordinate $XP = y$. The position of such points P is determined from the position of points p of the Sphere, according to some rule, but the position of p is determined by the two variables t and u, so the coordinates x and y can be considered Functions of each of the two variables t and u. Our inquiry then belongs to the part of Analysis which treats of Functions of two variables.

3. We now wish to take account the variation of the two quantities t and u in our calculation. To that end consider a point q on the Sphere, whose longitude $= t$ while its latitude $= u + du$. Moreover let r be a point with longitude $t + dt$ and latitude $l'r = u$. Completing the parallelogram $pqsr$, s will have longitude $t + dt$ and longitude $u + du$. Then on the Sphere the arc elements will be $pq = du$ and $ll' = dt$, so that the element $pr = dt \cdot \cos u$. Moreover, the parallelogram $pqrs$ will be a rectangle with diagonal

$$ps = \sqrt{du^2 + dt^2 \cdot \cos^2 u}.$$

4. Now suppose that to the points p,q,r,s on the Sphere correspond the points P,Q,R,S on the plane, and from these latter drop perpendiculars PX, QU, RV, and SW to the axis EF. Since between Q and P only the variable u changes, being incremented by du, the coordinates for the point Q will be

$$EU = x + du\left(\frac{dx}{du}\right), \quad \text{and} \quad UQ = y + du\left(\frac{dy}{du}\right).$$

In the same way, between P and R only the variable t changes, so that the coordinates for the point R will be

$$\text{abscissa} \quad EV = x + dt\left(\frac{dx}{dt}\right), \qquad \text{and ordinate} \quad VR = y + dt\left(\frac{dy}{dt}\right).$$

Then the point S, which is obtained from P by simultaneous changes in both t and u, will have for abscissa

$$EW = x + du\left(\frac{dx}{du}\right) + dt\left(\frac{dx}{dt}\right)$$

and for ordinate

$$WS = y + du\left(\frac{dy}{du}\right) + dt\left(\frac{dy}{dt}\right).$$

From this it appears that

$$XU = du\left(\frac{dx}{du}\right),$$

which is equal to the distance

$$VW = du\left(\frac{dx}{du}\right).$$

In the same way,

$$WS - VR = UQ - XP = du\left(\frac{dy}{du}\right). \tag{4.1}$$

But from this it follows that element $RS =$ element PQ, and similarly $PR = QS$, so that the quadrilateral $PQRS$ will in fact be a parallelogram.

5. Since the elementary rectangle $pqrs$ on the Sphere is represented on the plane by the parallelogram $PQRS$, let us begin by comparing the respective sides of the two figures. On the Sphere,

$$pq = du, \qquad \text{and} \qquad pr = dt \cdot \cos u,$$

while on the plane

$$PQ = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}, \qquad \text{and} \qquad PR = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Then clearly the element PQ corresponds to a small increment du in the direction of a Meridian, while PR corresponds to a small increment $dt \cos u$ in the direction of a Parallel[1]. Now if functions x and y could be obtained, satisfying

$$du = du\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}, \qquad \text{and} \qquad dt\cos u = dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

then lengths along both the Meridian and the Parallel would have the same size in the plane as on the Sphere. But there might be another, more significant, difference, namely, that the angle between them on the plane might differ from a right angle.

[1] Euler uses the capitalized term "Parallel" to refer to a circle of constant latitude on the sphere, and to the image of this circle on the plane.

6. So we begin by seeking the directions of the Meridian PQ and the Parallel PR with respect to the coordinates x and y. From Figure 2, the Meridian element PQ is inclined with our axis EF at an angle whose tangent is

$$\left(\frac{dy}{du}\right) : \left(\frac{dx}{du}\right).$$

In the same way, the direction of the Parallel PR is inclined with our axis EF at an angle whose tangent is

$$\left(\frac{dy}{dt}\right) : \left(\frac{dx}{dt}\right).$$

The difference between these two angles gives the angle, QPR, formed between the Parallel and the Meridian; its tangent is

$$\frac{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right) - \left(\frac{dx}{du}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{du}\right)\left(\frac{dx}{dt}\right) + \left(\frac{dy}{du}\right)\left(\frac{dy}{dt}\right)}.$$

In order that this angle be right, as it is on the Sphere, it is necessary that

$$\left(\frac{dx}{du}\right)\left(\frac{dx}{dt}\right)+\left(\frac{dy}{du}\right)\left(\frac{dy}{dt}\right) = 0, \quad \text{or} \quad \left(\frac{dy}{du}\right) : \left(\frac{dx}{du}\right) = -\left(\frac{dx}{dt}\right) : \left(\frac{dy}{dt}\right).$$

7. Therefore, if it is required that the plane figure $PQSR$ will be congruent to the Spherical figure $pqsr$, the following three conditions must be satisfied:

First, of course, that $PQ = pq$,
2) $PR = pr$,
3) angle $QPR = qpr = 90°$.

To this end, the following three equalities are required:

$$\text{I.} \quad \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} = 1, \qquad \text{or} \qquad \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = 1$$

$$\text{II.} \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \cos u, \qquad \text{or} \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cos^2 u$$

$$\text{III.} \quad \left(\frac{dy}{du}\right) : \left(\frac{dx}{dt}\right) = -\left(\frac{dx}{dt}\right) : \left(\frac{dy}{dt}\right).$$

If now we set

$$\left(\frac{dy}{du}\right) : \left(\frac{dx}{du}\right) = \operatorname{tang}\phi,$$

then by (III) it must be that

$$\left(\frac{dy}{dt}\right) : \left(\frac{dx}{dt}\right) = -\cot\phi,$$

and so

$$\left(\frac{dy}{du}\right) = \left(\frac{dx}{du}\right)\operatorname{tang}\phi \qquad \text{and} \qquad \left(\frac{dy}{dt}\right) = -\left(\frac{dx}{dt}\right)\cot\phi.$$

Substituting these values in the two first equations yields

$$\left(\frac{dx}{du}\right)^2 = \cos^2\phi \qquad \text{and} \qquad \left(\frac{dx}{dt}\right)^2 = \sin^2\phi\cos^2 u.$$

But evidently there are no circumstances under which the above three conditions can simultaneously be satisfied, since, as is well known, there is no way that the surface of a sphere can be represented exactly on a plane.

8. In order to remove differential expressions from the calculation, we make the following substitutions:

$$\left(\frac{dx}{du}\right) = p, \quad \left(\frac{dx}{dt}\right) = q, \quad \left(\frac{dy}{du}\right) = r, \quad \left(\frac{dy}{dt}\right) = s,$$

so that

$$dx = p\,du + q\,dt, \qquad dy = r\,du + s\,dt,$$

and before anything else, it is required that these two expressions be simultaneously integrable; this is the case if p, q, r, and s are functions of t and u such that

$$\left(\frac{dp}{dt}\right) = \left(\frac{dq}{du}\right) \qquad \text{and} \qquad \left(\frac{dr}{dt}\right) = \left(\frac{ds}{du}\right).$$

Moreover the above expressions for lengths of lines take the following form:

$$PQ = du\,\sqrt{pp + rr}, \qquad \text{and} \qquad PR = dt\,\sqrt{qq + ss}$$

Furthermore, the tangent of the angle of inclination of PQ with the axis is $= \frac{r}{p}$, the tangent of the angle of inclination of PR with the axis is $= \frac{s}{q}$, and finally, the tangent of angle QPR is

$$\frac{qr - ps}{pq + rs}.$$

9. Using this notation, it is required that a perfect mapping fulfill the following three conditions:

$$\text{I. } pp + rr = 1; \qquad \text{II. } qq + ss = \cos^2 u; \qquad \text{III. } \frac{r}{p} = -\frac{q}{s}.$$

We put

$$\frac{r}{p} = \operatorname{tang} \phi,$$

so that

$$\frac{s}{q} = -\cot \phi;$$

that is,

$$r = p \operatorname{tang} \phi, \qquad s = -q \cot \phi,$$

and the two first conditions yield

$$pp = \cos^2 \phi, \qquad qq = \sin^2 \phi \cos^2 u,$$

from which we deduce

$$p = \cos \phi, \qquad q = -\sin \phi \cos u$$

and thus

$$r = \sin \phi, \qquad s = \cos \phi \cdot \cos u.$$

Substituting these values into the expressions which must be integrable,

$$dx = du \cos \phi - dt \sin \phi \cos u$$
$$dy = du \sin \phi + dt \cos \phi \cos u;$$

From the requirements that

$$\left(\frac{dp}{dt}\right) = \left(\frac{dq}{du}\right), \quad \text{and} \qquad \left(\frac{dr}{dt}\right) = \left(\frac{ds}{du}\right),$$

arise the following two equalities:

$$\begin{aligned}
\text{I.}\qquad -\left(\frac{d\phi}{dt}\right)\sin\phi &= \sin u\sin\phi - \left(\frac{d\phi}{du}\right)\cos u\cos\phi,\\
\text{II.}\qquad \left(\frac{d\phi}{dt}\right)\cos\phi &= -\sin u\cos\phi - \left(\frac{d\phi}{du}\right)\cos u\sin\phi.
\end{aligned}$$

Combining these two by

$$(\text{I.})\times(\cos\phi)\quad+\quad(\text{II.})\times(\sin\phi)$$

yields

$$0=\left(\frac{d\phi}{du}\right)\cos u,\qquad\text{and thus}\qquad\left(\frac{d\phi}{du}\right)=0,$$

so that ϕ depends only on the variable t. But the combination

$$(\text{II.})\times(\cos\phi)\quad-\quad(\text{I.})\times(\sin\phi)$$

gives

$$\left(\frac{d\phi}{dt}\right)=-\sin u,$$

so that $\left(\frac{d\phi}{dt}\right)$ depends upon u, contradicting the previous result. Thus is demonstrated through computation that a perfect mapping of the Sphere onto the plane is not possible.

10. Since therefore a perfectly exact representation is excluded, we are obliged to admit representations which are not similar, so that the spherical figure differs in some manner from its image on the plane.

With respect to the divergence between the image and reality, we can make various assumptions, and according to the assumption which take as the basis, we can achieve the image most suitable for this or that purpose. In this way, the needs which the image must satisfy can vary in many different ways. From the infinite number of possibilities which present themselves, we shall in what follows discuss several which are especially important. Before all else, we assume that the angles formed by the Meridians and the Parallels are everywhere right angles, since if we admit acute and obtuse angles, the image produced will be competely useless. On this account, in what follows we shall always assume that the angle QPR is a right angle and therefore

$$\frac{r}{p} = -\frac{q}{s}.$$

11. We inquire more generally what can be deduced from the previous requirement, that all Parallels should cut the Meridians in right angles. To this end, we introduce again the angle ϕ, so that $r = p \operatorname{tang} \phi$ and thus $s = -q \cot \phi$. By substituting these values for r and s, the expressions for the two differential formulae which are to be integrated are rendered as

$$dx = p\,du + q\,dt, \qquad \text{and} \qquad dy = p\,du \operatorname{tang} \phi - q\,dt \cot \phi.$$

12. In order to put these expressions in the same form, we introduce in place of p and q two other variables, putting

$$p = m \cos \phi, \qquad q = n \sin \phi,$$

whence

$$r = m \sin \phi, \qquad s = -n \cos \phi,$$

and the two formulae to be integrated become

$$dx = m\,du \cos \phi + n\,dt \sin \phi,$$
$$dy = m\,du \sin \phi - n\,dt \cos \phi.$$

With this the entire task is reduced to the question: how should the functions m and n be selected, so that these two expressions are integrable? In addition, we must look back to the conditions we wish to fulfill, according to the case under consideration.

First Hypothesis

That all Meridians are set normal to our axis EF, while all Parallels are set parallel to it.

13. Since we supposed that $\operatorname{tang} \phi = \frac{r}{p}$, the angle ϕ measures the inclination of the arc element PQ with respect to the axis EF; furthermore, the direction

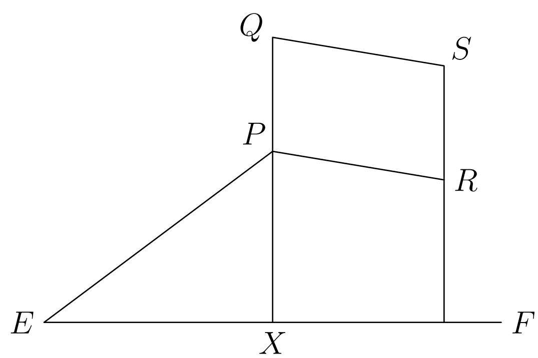

Figure 3: A copy of Euler's Figure 3. (See note 2, §15.)

PQ is that of the Meridian. and since the angle ϕ is assumed by hypothesis to be right, the two differential formulae become

$$dx = n\,dt, \qquad dy = m\,du.$$

That these become integrable, one can achieve in infinitely many ways; it is only necessary to take m as an arbitrary function of u, and n of t. For this reason it is possible to set additional conditions.

14. In the first place, all degrees of longitude can be made the same size; there is no reason to establish inequality among them. Thus, if our axis EF represents the Equator, so that the abscissa EX corresponds to the equatorial arc $al = t$, it is then possible to take $x = t$, in other words, to set the function n to unity, or to any constant, while for the ordinate one can take an arbitrary function of u.

15. Under such an hypothesis, not only does the parallelogram $PQSR$ become a rectangle, as on the Sphere, but also the point Q lies on the ordinate XP, so that $PQ = dy$ and $PR = dx = dt$ (Fig. 3).[2] If, furthermore, we set $y = u$, where u denotes the latitude of the place, then, if $dx = dt$ corresponds to a degree of longitude and $dy = du$ to a degree of latitude, we have that $dy = dx$; however, such a representation would be quite unusable, and all regions of the Earth would show severe distortion.

[2] In the original, PR and QS are drawn parallel to each other, but not parallel to EF, and $PQRS$ is drawn as a parallelogram, but not a rectangle. See the remark at the end of Note 3, p. 68, in [Wan1897]

16. It is better to take the ordinate y equal to some function u of the latitude, suitable to the purpose which the map is supposed to serve. A condition comes to mind, that the parallelogram $PQRS$ in the plane be similar to the parallelogram $pqrs$ on the Sphere, for then the smallest parts of the spherical surface will be similar to their images in the plane. It is exactly this condition on which the Maritime Charts, named after their inventor MERCATOR, are based.

I. On Mercator's Maritime Charts

16a.[3] Because it was required that the rectangle $PQRS$ be similar to the rectangle $pqrs$, in which $pq = du$ and $pr = \cos u\, dt$, it must be that

$$dy : dt = du : \cos u\, dt;$$

and since $dx = dt$,

$$dy = \frac{du}{\cos u};$$

integrating this expression gives

$$y = \ln \operatorname{tang}(45^\circ + \frac{1}{2}u).$$

The latitude, measured on the Sphere by the angle u, thus corresponds on the image to the hyperbolic logarithm of the tangent of the angle $45^\circ + \frac{1}{2}u$. Using this formula, tables are constructed in which for specific values of u the corresponding values of y are recorded.

17. Given that on the map all Parallels are equal to the Equator, while on the Sphere they become smaller and smaller, it follows that in this representation the degrees of each Meridian, which are equal on the Sphere, must become larger at the same rate as the degrees on each Parallel increaase with respect to the Sphere. For this reason, the degrees of latitude on a Meridian increase constantly as the latitude becomes greater, and at the same rate as the cosine of the latitude decreases. Therefore, if du denotes a degree of a Meridian on

[3] Euler's original paper contained two paragraphs numbered "16". Following both [SSN1894] and [Wan1897], and to keep numbering consistent for those who wish to consult the original text, we call the second of these "16a". See also Note 5, p. 68 in [Wan1897]

the Sphere, then on the Map, the length of the same degree will be $\dfrac{du}{\cos u}$. For example, at the latitude 60° one degree on the Meridian has twice the length as on the sphere, and at the pole it becomes infinitely long. Hence, such maps can never be extended to the poles.

18. The greatest advantage which this Map gives to travelers at sea, is that the Loxodromic curves, which on the Sphere cut each Meridian at the same angle, are here represented by a straight line. Such a straight line cuts all the Meridians, which are parallel to each other, at the same angle.

19. If, for example, the line ap on the Sphere [Fig. 1] refers to a Loxodromic curve which cuts every Meridian at an angle $= \zeta$, and if the length of $ap = z$, then

$$du : dz = \cos\zeta : 1, \quad \text{so that} \quad dz = \frac{du}{\cos\zeta} \quad \text{and} \quad z = \frac{u}{\cos\zeta}.$$

But if, on the plane, the line EP [Fig. 3] corresponds to ap, then the angle EPX is also $= \zeta$, and clearly EP is a straight line with length $\dfrac{y}{\cos\zeta}$. If the length of line EP is known, then, inversely, one can obtain from this the length of the path traversed by the ship, that is, the length of ap, since

$$ap : EP = u : y,$$

and the ratio $u : y$ can be considered as known.

20. But while Loxodromic curves are represented on the plane simply as straight lines, in constrast, great circles on the Sphere are represented by transcendental curves of a very high order. Let ap [Fig. 1] be the arc of a great circle inclined to the Equator at point a by an angle of $lap = \theta$. It is well known that

$$\operatorname{tang} u = \operatorname{tang}\theta \, \sin t;$$

with this and the two previous formulae,

$$x = t, \quad \text{and} \quad y = \ln \operatorname{tang}(45^\circ + \tfrac{1}{2}u),$$

the curve EP, which corresponds to the arc ap, can be defined.

21. In order to determine the nature of the curve under consideration, denote by e the number whose hyperbolic logarithm $= 1$; then

$$e^y = \operatorname{tang}\left(45^\circ + \tfrac{1}{2}u\right) = \frac{1 + \operatorname{tang}\frac{1}{2}u}{1 - \operatorname{tang}\frac{1}{2}u},$$

thus

$$\operatorname{tang}\tfrac{1}{2}u = \frac{e^y - 1}{e^y + 1};$$

from which, collecting terms again,

$$\operatorname{tang} u = \frac{e^{2y} - 1}{2e^y}.$$

Substituting this value for $\operatorname{tang} u$ in the equation and considering that $t = x$, the following relation between x and y is produced:

$$\frac{e^{2y} - 1}{2e^x} = \operatorname{tang}\theta \sin x,$$

by which is expressed the nature of the curve EF. From this last it can be inferred that if x is very small, then y becomes very small also. For very small values of y, we have

$$e^y = 1 + y \quad \text{and} \quad e^{2y} = 1 + 2y;$$

whence, if $\sin x = x$, we obtain

$$\frac{y}{1 + y} = x \operatorname{tang}\theta$$

or just

$$\frac{y}{x} = \operatorname{tang}\theta,$$

so that at point E the curve is inclined to the Equator with an angle of θ.

Next, if we take $x = 90^\circ$, then

$$\frac{e^{2y} - 1}{2e^y} = \operatorname{tang}\theta,$$

whence it follows that

$$e^y = \operatorname{tang}\theta \pm \sqrt{\operatorname{tang}^2\theta + 1} = \frac{\sin\theta + 1}{\cos\theta} = \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \cot(45^\circ - \tfrac{1}{2}\theta),$$

and thus

$$y = \ln\cot\left(45^\circ - \tfrac{1}{2}\theta\right) = \ln\operatorname{tang}\left(45^\circ + \tfrac{1}{2}\theta\right).$$

From this it can be seen that the curve is transcendental of a very high order.

II. ON MAPS IN WHICH EVERY SURFACE AREA IS REPRESENTED AT ITS TRUE SIZE

22. We continue to assume that all Meridians are parallel to each other, and that all degrees are equal at the Equator. The degrees of longitude, calculated along any Parallel, have then the same magnitude as at the Equator, so once again we take $x = t$. Now it is required that the area of the rectangle $PQRS = dx\,dy$ is equal to that of the rectangle $pqrs$ on the Sphere $= du\,dt\,\cos u$. For this it is only necessary that $dy = \cos u\,du$, from which we obtain by integration $y = \sin u$. It is then very easy to construct a map; one merely makes the ordinates equal to the sine of the corresponding latitudes. The degrees of latitude along the Meridians become smaller and smaller as the the distance from the Equator increases, and vanish entirely at the pole. The pole itself is represented by a straight line, parallel to the Equator EF, and a distance equal to 1 from the latter; this distance is equal to the radius of the Sphere.

23. If the entire surface of the Earth be represented by this method, the map will have the form of a rectangle whose length will be equal to the circumference of the equator $= 2\pi$; on each side of the Equator, the distance in latitude to the pole is $= 1$, and thus, the area of the rectangle will be $= 4\pi$; that is, equal to the area of the entire spherical surface. In such maps, all countries of the Earth are represented at their true size, although their shape displays a great deviation from reality. In such a representation, the area of any region on the map has the is equal to the area of the same region on the surface of the Earth. Such maps can be used to compare the true size of different regions of the Earth. This is best accomplished by the use of units such as square degrees, where one degree along the Equator is reckoned as fifteen German miles[4]

[4] A note from [Bag1959] states that one German mile was equal to 7.4204 km. This puts the length of one degree along the equator at 111.306 km, which is very close to the

SECOND HYPOTHESIS

That small regions of the Earth should be displayed as similar figures in the plane.

24. In order that such a similitude be observed, it is necessary before all else that Meridians and Parallels are everywhere perpendicular. For this reason the two differential formulae which must be integrabile, encountered above in paragraph 12, are expressed as

$$dx = m\,du\cos\phi + n\,dt\sin\phi$$
$$dy = m\,du\sin\phi - n\,dt\cos\phi.$$

Furthermore,

$$PQ = du\sqrt{pp+rr} = m\,du,$$
$$PR = dt\sqrt{qq+ss} = n\,dt.$$

and QPR is a right angle, in accordance with the previously established formulae.

25. If the rectangle $PQRS$ is similar to $pqrs$, then it is necessary that $PQ : PR = pq : pr$, that is, $m : n = 1 : \cos u$, so that our two differential formulae become:

$$dx = m\,du\cos\phi + m\,dt\cos u\sin\phi,$$
$$dy = m\,du\sin\phi - m\,dt\cos u\cos\phi.$$

26. The entire problem is thus reduced to the following: which functions of t and u must one take for m and ϕ, so that both of the differential formulae are integrable? For the sake of brevity, we introduce again p and r in place of m and ϕ As above

$$p = m\cos\phi, \quad \text{and} \quad r = m\sin\phi;$$

currently accepted value of 111.324 km.

we have thus the equations

$$dx = p\,du + r\,dt\cos u,$$
$$dy = r\,du - p\,dt\cos u,$$

and we ask which functions of t and u is it necessary to take for p and r, so that both of these equations are integrable. A solution to this problem is obtained, as can easily be seen, in the case of the Nautical Chart. In this case, it is only necessary to take

$$p = 0, \quad r = \frac{1}{\cos u}.$$

Other solutions, however, are not so easily found.

27. From well-known conditions for integrability, it is required that:

$$\begin{aligned} \left(\frac{dp}{dt}\right) &= \left(\frac{dr\ \cos u}{du}\right) = -r\sin u + \cos u\left(\frac{dr}{du}\right), \\ \left(\frac{dr}{dt}\right) &= -\left(\frac{dp\ \cos u}{du}\right) = p\sin u - \cos u\left(\frac{dp}{du}\right). \end{aligned}$$

From the latter it follows that

$$\left(\frac{dp}{du}\right) = p\ \mathrm{tang}\,u - \left(\frac{dr}{dt}\right)\frac{1}{\cos u},$$

and since

$$dp = du\left(\frac{dp}{du}\right) + dt\left(\frac{dp}{dt}\right),$$

the following new condition arises:

$$dp = p\,du\,\mathrm{tang}\,u - \left(\frac{dr}{dt}\right)\frac{du}{\cos u} - r\,dt\sin u + \left(\frac{dr}{du}\right)dt\cos u.$$

On multiplying by $\cos u$ and bringing the term in p to the left side, this becomes

$$dp\cos u - p\,du\sin u = -r\,dt\sin u\cos u + \left(\frac{dr}{du}\right)dt\cos^2 u - \left(\frac{dr}{dt}\right)du.$$

In order that the left side of this equation be integrable the right hand must be also, and for r a suitable function of t and u must be sought.

28. It is now necessary to find a way to resolving these formulae. After careful consideration of all the difficulties, two methods of reaching the goal presented themselves to me. One of these gives an infinite number of particular solutions, while the other has led me to the most general solution. I shall develop both methods here, so that through them significant advances in the Analysis of functions of two variables might be obtained.

Method of locating particular solutions to the differential equations

$$dx = p\,du + r\,dt\cos u, \qquad dy = r\,du - p\,dt\cos u$$

29. Since the functions p and r each involve the two variables u and t, we set each equal to the product of a certain function of u by a certain function of t. Thus, let

$$p = U\,T, \qquad \text{and} \qquad r = V\,\Theta,$$

where U and V are functions of u only, T and Θ functions of t only. We then have the two differential formulae rendered as:

$$\begin{aligned} I.\quad dx &= UT\,du \;+\; V\Theta\,dt\cos u \\ II.\quad dy &= V\Theta\,du \;-\; UT\,dt\cos u. \end{aligned}$$

30. From this it will be possible to produce a double representation of x and y in the form of an integral. If the quantity t is taken as constant, the second terms vanish, and from the first are obtained:

$$x = T\int U\,du, \qquad \text{and} \qquad y = \Theta\int V\,du.$$

If, on the other hand, u is taken as constant, then from the second terms come

$$x = V\cos u\int \Theta\,dt, \qquad \text{and} \qquad y = -U\cos u\int T\,dt.$$

Moreover, the two expressions for x must equal each other; likewise the two expressions for y:

$$\begin{aligned} T\int U\,du &= V\cos u\int \Theta\,dt, &\quad \text{or}\quad \frac{\int U\,du}{V\cos u} &= \frac{\int \Theta\,dt}{T} \\ \Theta\int V\,du &= -U\cos u\int T\,dt, &\quad \text{or}\quad \frac{\int V\,du}{U\cos u} &= -\frac{\int T\,dt}{\Theta}. \end{aligned}$$

From these two expressions the functions U, V, T, Θ must be determined.

31. If it must be that

$$\frac{\int U\,du}{V\cos u} = \frac{\int \Theta\,dt}{T},$$

then clearly both fractions must be equal to a fixed quantity; furthermore, the two variables t and u are independent of each other. Denoting the constant by α,

$$\int U\,du = \alpha\,V\cos u, \qquad \text{and} \qquad \int \Theta\,dt = \alpha T.$$

In the same way, if

$$\frac{\int V\,du}{U\cos u} = -\frac{\int T\,dt}{\Theta},$$

then each of the fractions must equal some constant β and therefore

$$\int V\,du = \beta\;U\cos u, \qquad \text{and} \qquad \int T\,dt = -\beta\;\Theta.$$

Thereby are the integrals appearing in the formulae reduced to absolute magnitudes, and the values for x and y may be expressed without integration signs:

$$x = \alpha\,T\,V\;\cos u \qquad \text{and} \qquad y = \beta\,\Theta\,U\;\cos u.$$

32. To abbreviate, let us set $U\;\cos u = P$ and $V\;\cos u = Q$, so that

$$U = \frac{P}{\cos u}, \quad V = \frac{Q}{\cos u}.$$

Our four formulae then become

$$\int \Theta\, dt = \alpha\, T \qquad \text{and} \qquad \int T\, dt = -\beta\Theta,$$
$$\int \frac{P\, du}{\cos u} = \alpha\, Q \qquad \text{and} \qquad \int \frac{Q\, du}{\cos u} = \beta\, P.$$

Taking derivatives of the first two of these expressions gives

$$\Theta = \alpha \frac{dT}{dt}, \qquad P = \frac{\alpha\, dQ \cos u}{du},$$

and substituting these into the latter two gives

$$\int T\, dt = -\frac{\alpha\beta\, dT}{dt}, \quad \text{and} \quad \int \frac{Q\, du}{\cos u} = \frac{\alpha\beta\, dQ \cos u}{du}.$$

Again taking derivatives of these expressions, by t and by u respectively:

$$T = -\frac{\alpha\beta\, ddT}{dt^2}, \qquad \text{and} \qquad Q = \frac{\alpha\beta\, ddQ \cos^2 u}{du^2} - \frac{\alpha\beta\, dQ \sin u \cos u}{du}.$$

We have now established two second-order differential equations, upon whose integration depends the solution of our problem.

34. [5] Let us begin with the first equation,

$$T = -\frac{\alpha\beta\, ddT}{dt^2}.$$

Multiplying by $2\, dT$ and integrating produces

$$TT = -\frac{\alpha\beta\, dT^2}{dt^2} + A;$$

from which, collecting terms,

$$dt^2 = \frac{\alpha\beta\ dT^2}{A - TT}.$$

[5] The original edition did not contain a paragraph 33.

Proceeding in the same way with the second equation,

$$Q = \frac{\alpha\beta\, dd\,U \cos^2 u}{du^2} - \frac{\alpha\beta\, dQ \sin u \cos u}{du},$$

multiplying by $2\,dQ$ and integrating produces

$$QQ = \frac{\alpha\beta\, dQ^2 \cos^2 u}{du^2} + B;$$

from which, collecting terms,

$$\frac{du^2}{\cos^2 u} = \frac{\alpha\beta\, dQ^2}{QQ - B}.$$

For the final Integration, we must distinguish two cases, according to whether the quantity $\alpha\beta$ be positive or negative.

First Case

which is $\alpha\beta = +\lambda\lambda$, *and thus* $\beta = +\dfrac{\lambda\lambda}{\alpha}$.

35. In this case we have

$$dt^2 = \frac{\lambda\lambda\, dT^2}{A - TT},$$

where, since A must be a positive quantity, we can write $A = aa$. Thus

$$dt = \frac{\lambda\, dT}{\sqrt{aa - TT}},$$

whose integral is clearly

$$t + \delta = \lambda \arcsin\left(\frac{T}{a}\right),$$

so that, collecting terms,

$$T = a \sin\left(\frac{t + \delta}{\lambda}\right),$$

whence

$$dT = \frac{a\,dt}{\lambda} \cos\left(\frac{t+\delta}{\lambda}\right).$$

From $\Theta = \alpha\,\dfrac{dT}{dt}$ it now follows that

$$\Theta = \frac{a\alpha}{\lambda} \cos\left(\frac{t+\delta}{\lambda}\right).$$

36. The other equation to be integrated becomes, for the case $\alpha\beta = \lambda\lambda$,

$$\frac{du}{\cos u} = \frac{\lambda\,dQ}{\sqrt{QQ-B}};$$

which, being integrated, gives

$$\ln \operatorname{tang}\left(45^\circ + \frac{1}{2}u\right) + \lambda\,\ln\varepsilon = \lambda\,\ln\left(Q + \sqrt{QQ-B}\right).$$

So that we are able more conveniently to develop this formula, we call

$$\operatorname{tang}(45^\circ + \frac{1}{2}u) = s,$$

and since

$$\ln s = \int \frac{du}{\cos u},$$

we have

$$\frac{ds}{s} = \frac{du}{\cos u}, \qquad \text{and thus} \qquad ds = \frac{s\,du}{\cos u}.$$

Therefore,

$$\ln(\varepsilon^\lambda s) = \lambda\,\ln\left(Q + \sqrt{QQ-B}\right),$$

from which it follows that

$$\varepsilon^\lambda s = (Q + \sqrt{QQ-B})^\lambda,$$

and so

$$Q+\sqrt{QQ-B}=\varepsilon s^{\frac{1}{\lambda}}.$$

Abbreviating $\frac{1}{\lambda}=\nu$ and solving for Q gives

$$Q=\frac{1}{2}\varepsilon s^{\nu}+\frac{Bs^{-\nu}}{2\varepsilon},$$

and from this comes

$$dQ=\frac{1}{2}\nu\varepsilon s^{\nu-1}\,ds-\frac{\nu B}{2\varepsilon}s^{-\nu-1}\,ds;$$

from the equality $ds=\dfrac{s\,du}{\cos u}$, this changes into

$$dQ==\frac{\frac{1}{2}\nu\varepsilon s^{\nu}}{\cos u}-\frac{\nu B}{2\varepsilon}s^{-\nu}\frac{du}{\cos u}.$$

Since[6]

$$P=\frac{\alpha\,dQ\cos u}{du},$$

we now have

$$P=\frac{1}{2}\,\alpha\,\nu\,\varepsilon s^{\nu}-\frac{\alpha\,\nu\,Bs^{-\nu}}{2\varepsilon}.$$

[6]From §32

37. From the values found above for P and Q, it follows that

$$U = \frac{\alpha\nu\varepsilon s^{\nu}}{2\cos u} - \frac{\alpha\nu B s^{-\nu}}{2\varepsilon\cos u}, \qquad \text{and} \qquad V = \frac{\varepsilon s^{\nu}}{2\cos u} + \frac{B s^{-\nu}}{2\varepsilon\cos u};$$

from which, finally, we obtain both our coordinates x and y:[7]

$$x = \frac{1}{2}\alpha\, a \sin\left(\frac{t+\delta}{\lambda}\right)\left(\varepsilon s^{\nu} + \frac{B}{\varepsilon}s^{-\nu}\right),$$
$$y = \frac{1}{2}\alpha\, \nu\, \lambda\, a \cos\left(\frac{t+\delta}{\lambda}\right)\left(\varepsilon s^{\nu} - \frac{B}{\varepsilon}s^{-\nu}\right),$$

recalling that

$$\nu = \frac{1}{\lambda} \qquad \text{and} \qquad s = \text{tang}(45^\circ + \frac{1}{2}u).$$

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[7]Perhaps it would be useful to summarize Euler's calculations for x and y. From §32, $U\cos u = P$ and $V\cos u = Q$. From this, and the expressions for P and Q obtained in §36,

$$U = \frac{\alpha\nu\varepsilon s^{\nu}}{2\cos u} - \frac{\alpha\nu B s^{-\nu}}{2\varepsilon\cos u}, \qquad \text{and} \qquad V = \frac{\varepsilon s^{\nu}}{2\cos u} + \frac{B s^{-\nu}}{2\varepsilon\cos u};$$

From §30, §31 and §35, respectively,

$$x = V\cos u \int \Theta\, dt, \qquad \int \Theta\, dt = \alpha T, \qquad T = a\sin\left(\frac{t+\delta}{\lambda}\right)$$

so that

$$x = (\alpha T)(V\cos u) = \alpha a \sin\left(\frac{t+\delta}{\lambda}\right)\left(\frac{\varepsilon s^{\nu}}{2\cos u} + \frac{B s^{-\nu}}{2\varepsilon\cos u}\right).$$

To calculate y, recall that $\alpha\beta$ is equal to a positive constant λ^2; therefore $\beta = \frac{\lambda^2}{\alpha}$. From §30, §31, and §35, respectively

$$y = \Theta \int V\, du, \qquad \int V\, du = \beta\, U\cos u, \qquad \Theta = \frac{a\alpha}{\lambda}\cos\left(\frac{t+\delta}{\lambda}\right)$$

so that

$$y = \frac{a\alpha}{\lambda}\cos\left(\frac{t+\delta}{\lambda}\right)\cdot\beta\cdot U\cos u = \frac{a\alpha}{\lambda}\cos\left(\frac{t+\delta}{\lambda}\right)\cdot\frac{\lambda^2}{\alpha}\cdot\cos u\left(\frac{\alpha\nu\varepsilon s^{\nu}}{2\cos u} - \frac{\alpha\nu B s^{-\nu}}{2\varepsilon\cos u}\right).$$
$$= a\lambda\cos\left(\frac{t+\delta}{\lambda}\right)\cdot\tfrac{1}{2}\alpha\nu\left(\varepsilon s^{\nu} - \frac{B}{\varepsilon}s^{-\nu}\right),$$

These formulae are rendered more elegantly, if we set $B = \epsilon^2 b$. From this we obtain

$$x = \frac{1}{2}\alpha\,\varepsilon\,a\,\sin\left(\frac{t+\delta}{\lambda}\right)\left(s^{\frac{1}{\lambda}} + s^{-\frac{1}{\epsilon}}\right),$$
$$y = \frac{1}{2}\alpha\,\varepsilon\,a\,\cos\left(\frac{t+\delta}{\lambda}\right)\left(s^{\frac{1}{\lambda}} - s^{-\frac{1}{\epsilon}}\right).$$

Second Case

where $\alpha\beta = -\mu\mu$, and thus $\beta = -\dfrac{\mu\mu}{\alpha}$.

38. In this case, we have

$$d\,t^2 = \frac{-\mu\mu\,dT^2}{A - TT},$$

and from this,

$$d\,t = \frac{-\mu\,dT}{\sqrt{TT - A}}.$$

From which, integrating,

$$t + \delta = \mu \ln\left(T + \sqrt{TT - A}\right),$$

from which, if e denotes the number whose hyperbolic logarithm is 1,

$$e^{(t+\delta)/\mu} = T + \sqrt{TT - A}.$$

For the sake of brevity let $\dfrac{t+\delta}{\mu} = \theta$, so that $d\theta = \dfrac{dt}{\mu}$; then

$$e^{\theta} - T = \sqrt{TT - A},$$

whence

$$T = \frac{e^{2\theta} + A}{2e^{\theta}} = \tfrac{1}{2}e^{\theta} + \tfrac{1}{2}Ae^{-\theta}.$$

But from this,

$$dT = \frac{dt}{2\mu}e^{\theta} - \frac{A\,dt}{2\mu}e^{-\theta},$$

so that[8]

$$\Theta = \frac{\alpha}{2\mu}(e^{\theta} - Ae^{-\theta}).$$

39. Moreover, in this case, we have

$$\frac{du^2}{\cos^2 u} = \frac{-\mu\mu\, dQ^2}{QQ - B} = \frac{\mu\mu\, dQ^2}{B - QQ}.$$

Since the quantity B must be positive, we set $B = bb$, so that

$$\frac{du}{\cos u} = \frac{\mu\, dQ}{\sqrt{b\, b - QQ}},$$

and integrating,

$$\ln \operatorname{tang}(45^\circ + \tfrac{1}{2}u) + \ln \varepsilon = \mu \cdot \arcsin\left(\frac{Q}{b}\right);$$

where if we again set $\operatorname{tang}(45^\circ + \frac{1}{2}u) = s$, then

$$\frac{\ln(\varepsilon s)}{\mu} = \arcsin\left(\frac{Q}{b}\right),$$

from which in turn we deduce

$$Q = b\, \sin\left(\frac{\ln(\varepsilon s)}{\mu}\right),$$

and from this

$$dQ \;=\; \frac{b}{\mu} \cdot \frac{ds}{s}\, \cos\left(\frac{\ln(\varepsilon s)}{\mu}\right) \;=\; \frac{b}{\mu} \cdot \frac{du}{\cos u}\, \cos\left(\frac{\ln(\varepsilon s)}{\mu}\right),$$

so that, finally,

$$P = \frac{\alpha b}{\mu} \cos\left(\frac{\ln(\varepsilon s)}{\mu}\right).$$

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[8]See §35

40. Now from the above[9]

$$x = \alpha TV \cos u = \alpha\, TQ, \qquad \text{and} \qquad y = \beta\, \Theta P = -\frac{\mu\mu}{\alpha}\Theta P$$

which become, by substituting the values discovered,

$$x = \tfrac{1}{2}\,\alpha\, b\, \sin\frac{\ln(\varepsilon s)}{\mu}\,\left(e^{\theta} + Ae^{-\theta}\right)$$

and

$$y = \tfrac{1}{2}\,\alpha\, b\, \cos\frac{\ln(\varepsilon s)}{\mu}\,\left(e^{\theta} - Ae^{-\theta}\right),$$

where it must be remembered that

$$\theta = \frac{t+\delta}{\mu} \qquad \text{and} \qquad s = \operatorname{tang}(45^\circ + \tfrac{1}{2}u).$$

41. The above formulae contain several completely arbitrary quantities, so that these solutions can be extended to embrace innumerable speccial cases. However, we can obtain a still more general solution, if we compound two, or arbitrarily many, solutions in the above form. That is to say, if we have first found the values $x = M$, $y = N$, and then $x = M'$, $y = N'$, and then $x = M''$, $y = N''$, etc., than the following very general solution can be constructed:

$$\begin{aligned} x &= \mathfrak{A}M + \mathfrak{B}M' + \mathfrak{C}M'' + \mathfrak{D}M''' + \cdots, \\ y &= \mathfrak{A}N + \mathfrak{B}N' + \mathfrak{C}N'' + \mathfrak{D}N''' + \cdots; \end{aligned}$$

[9] From §30 and §31,

$$x = \alpha\, T\int U\, du, \qquad \int U\, du = \alpha V \cos u;$$

from §30, §31, and §32,

$$y = \Theta\int V\, du, \qquad \int V\, du = \beta\cos u, \qquad U\cos u = P.$$

and certainly this method is so general that it contains all possible solutions.

General Method of Resolving the Differential Equations

$$dx = p\,du + r\,dt\cos u, \qquad dy = r\,du - p\,dt\cos u$$

42. What is sought for is some combination of the two formulae, which admits a resolution into two factors. To this end, multiply the first by α, the second by β, and add to obtain

$$\alpha\,dx + \beta\,dy = p(\alpha\,du - \beta\,dt\cos u) + r(\beta\,du + \alpha\,dt\cos u);$$

in order to bring the two differential factors to the same form, this can be written as

$$\alpha\,dx + \beta\,dy = \alpha p\,(du - \frac{\beta}{\alpha}\,dt\cos u) + \beta r(du + \frac{\alpha}{\beta}\,dt\cos u).$$

Now we take

$$\frac{\alpha}{\beta} = -\frac{\beta}{\alpha}, \qquad \text{or} \qquad \alpha\alpha + \beta\beta = 0, \qquad \text{or} \qquad \beta = \alpha\sqrt{-1},$$

and the combination gives

$$dx + dy\sqrt{-1} = (p + r\sqrt{-1})(du - \sqrt{-1}\,dt\cos u).$$

In order that the differential factor on the right can be integrated, this will be represented in the form

$$dx + dy\sqrt{-1} = \cos u(p + r\sqrt{-1})(\frac{du}{\cos u} - \sqrt{-1}\,dt).$$

43. We now set

$$\frac{du}{\cos u} - dt\sqrt{-1} = dz,$$

so that

$$z = \ln \operatorname{tang}\left(45^\circ + \tfrac{1}{2}u\right) - t\sqrt{-1},$$

and

$$dx + dy\sqrt{-1} = \cos u(p + r\sqrt{-1})\, dz.$$

Clearly, this equation is not integrable unless the factor on the right,

$$\cos u(p + r\sqrt{-1}),$$

is itself a function of z; and whatever function it be, integration can always be performed. From this it follows that the the integral is also a function of z, so that the expression $x + y\sqrt{-1}$ equals an arbitrary function of z, that is, of the quantity

$$\ln \operatorname{tang}\left(45^\circ + \tfrac{1}{2}\right) - t\sqrt{-1}.$$

44. In order to render the formula more elegant, we set, as previously,

$$\operatorname{tang}\left(45^\circ + \tfrac{1}{2}u\right) = s,$$

so that

$$\frac{ds}{s} = \frac{du}{\cos u}, \quad \text{and} \quad z = \ln s - t\sqrt{-1}.$$

Denote, as is customary, by Γ an arbitrary function of its argument,

$$x + y\sqrt{-1} = \Gamma : (\ln s - t\sqrt{-1}),$$

or rather, what comes out the same,

$$x + y\sqrt{-1} = 2\Gamma : (\ln s - t\sqrt{-1}).$$

Since the expression $\sqrt{-1}$ by its nature has the double sign $\pm$, we also have

$$x - y\sqrt{-1} = 2\Gamma : (\ln s + t\sqrt{-1}).$$

Thus we infer that

$$\begin{aligned} x &= \Gamma : (\ln s - t\sqrt{-1}) + \Gamma : (\ln s + t\sqrt{-1}) \\ y\sqrt{-1} &= \Gamma : (\ln s - t\sqrt{-1}) - \Gamma : (\ln s + t\sqrt{-1}) \end{aligned}$$

However, it is certain that these expressions for x and y can always be reduced to real values.

45. For example, if Γ represents any power of the argurment, or any multiple, then, denoting the exponent by γ, and using the abbreviation $\ln s = v$, development of the power series yields

$$x = v^{\lambda} - \frac{\lambda(\lambda-1)}{1\cdot 2}v^{\lambda-2}tt + \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{1\cdot 2\cdot 3\cdot 4}v^{\lambda-4}t^4 - \frac{\lambda\cdots(\lambda-5)}{1\cdot 2\cdots 6}v^{\lambda-6}t^6 + \text{etc.}$$
$$y = \frac{\lambda}{1}v^{\lambda-1}t - \frac{\lambda(\lambda-1)(\lambda-2)}{1\cdot 2\cdot 3}v^{\lambda-3}t^3 + \frac{\lambda\cdots(\lambda-4)}{1\cdot 2\cdots 5}v^{\lambda-5}t^5 - \frac{\lambda\cdots(\lambda-6)}{1\cdot 2\cdots 7}v^{\lambda-7}t^7 + \text{etc.}$$

In fact, the value of y needed to receive opposite signs, but one can, depending on the nature of the object, transpose the positive and negative directions of the axes.

46. These values are apparently quite different from those which were obtained above from our particular solutions. On the other hand, they are immediately valid for the case of Nautical Charts, which was not contained in the formulae derived above. One only need set $\lambda = 1$, so that

$$x = \ln s = \ln \text{tang}\left(45^\circ + \tfrac{1}{2}u\right) \quad \text{and} \quad y = t.$$

Indeed, in the above, the values of x and y were exchanged, but clearly the x and y coordinates can always be permuted.

47. While it is certain that all the values found above must be contained within the current formula, since the latter clearly constitutes the most general solution, it is well worth the trouble to show that this is really the case. To this end, observe that if $\Gamma : z$ denotes an arbitrary function of z, it is always possible to write in its place $\Delta : Z$, Z being itself a certain function of z. Here we take $Z = e^{\alpha z}$, with $z = \ln s - t\sqrt{-1}$; then in place of $\Gamma : (\ln s - t\sqrt{-1})$ it is possible to write

$$\Delta : e^{\alpha \ln s - t\sqrt{-1}}$$

. But since $e^{\alpha \ln s} = s^{\alpha}$, and

$$e^{\alpha t\sqrt{-1}} = \cos\alpha t + \sqrt{-1}\,\sin\alpha t,$$

we have

$$e^{\alpha \ln s\, -\, \alpha t\sqrt{-1}} = s^{\alpha}\left(\cos\alpha t - \sqrt{-1}\sin\alpha t\right).$$

Replacing Γ with the new function Δ, the formulae in §44 become

$$x = \Delta : s^{\alpha}\left(\cos\,\alpha t - \sqrt{-1}\sin\alpha t\right)\ +\ \Delta : s^{\alpha}\left(\cos\,\alpha t + \sqrt{-1}\sin\alpha t\right),$$
$$y\sqrt{-1} = \Delta : s^{\alpha}\left(\cos\,\alpha t - \sqrt{-1}\sin\alpha t\right)\ -\ \Delta : s^{\alpha}\left(\cos\,\alpha t + \sqrt{-1}\sin\alpha t\right);$$

where it is observed that both values can be multiplied by an arbitrary constant; and also they can be interchanged.

48. Considering the special case $\Delta : z = z$, we obtain

$$x = 2s^{\alpha}\ \cos\alpha t, \quad \text{and} \qquad\qquad y = -2s^{\alpha}\sin\alpha t.$$

If α is taken in a negative sense, the following values also satisfy:

$$x = 2s^{-\alpha}\cos\alpha t, \qquad\qquad y = +2s^{-\alpha}\sin\alpha t.$$

As observed above, that the two solutions can always be combined in such a manner that both are multiplied by a constant and then added together. Thus, out of the two previous solutions the more general solution can be formed:

$$x = \left(\mathfrak{A}s^{\alpha} + \mathfrak{B}s^{-\alpha}\right)\cos\alpha t, \qquad y = \left(-\mathfrak{A}s^{\alpha} + \mathfrak{B}s^{-\alpha}\right)\sin\alpha t.$$

The solution given in §37 is included in these formulae. But clearly the formulae which include the function Δ are much more general.

49. In order to derive the second particular solution [§40] from our general formula, we set

$$\begin{aligned} Z &= \cos\alpha z = \cos\left(\alpha\ln s - \alpha t\sqrt{-1}\right) \\ &= \cos(\alpha\ln s)\cos(\alpha t\sqrt{-1}) + \sin(\alpha\ln s)\sin(\alpha t\sqrt{-1}). \end{aligned}$$

But it is well known that

$$\cos(\alpha t\sqrt{-1}) = \frac{e^{-\alpha t} + e^{+\alpha t}}{2}$$

and

$$\sin(\alpha t\sqrt{-1}) = \frac{e^{-\alpha t} - e^{+\alpha t}}{2\sqrt{-1}},$$

so that

$$Z = \left(\frac{e^{-\alpha t}+e^{+\alpha t}}{2}\right)\cos(\alpha\ln s) \quad + \quad \left(\frac{e^{-\alpha t}-e^{+\alpha t}}{2\sqrt{-1}}\right)\sin(\alpha\ln s).$$

From this we have

$$\begin{aligned} x = \Delta : &\left(\frac{\cos(\alpha\ln s)(e^{-\alpha t}+e^{+\alpha t})}{2}+\frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{2\sqrt{-1}}\right)\\ &+\Delta : \left(\frac{\cos(\alpha\ln s)(e^{-\alpha t}+e^{+\alpha t})}{2}-\frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{2\sqrt{-1}}\right), \end{aligned}$$

and

$$\begin{aligned} y\sqrt{-1} = \Delta : &\left(\frac{\cos(\alpha\ln s)(e^{-\alpha t}+e^{+\alpha t})}{2}+\frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{2\sqrt{-1}}\right)\\ &-\Delta : \left(\frac{\cos(\alpha\ln s)(e^{-\alpha t}+e^{+\alpha t})}{2}-\frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{2\sqrt{-1}}\right). \end{aligned}$$

But then, selecting $\Delta : Z$ to be Z, this becomes:

$$x = \cos(\alpha\ln s)(e^{-\alpha t}+e^{+\alpha t}), \qquad y\sqrt{-1} = \frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{\sqrt{-1}},$$

and in the case where α is negative:

$$x = \cos(\alpha\ln s)(e^{+\alpha t}+e^{-\alpha t}), \qquad y\sqrt{-1} = -\frac{\sin(\alpha\ln s)(e^{-\alpha t}-e^{+\alpha t})}{\sqrt{-1}}.$$

These formulae include the solution, presented in §40, of the second case.

50. In these general formulae for finding the coordinates x and y are included all possible representations of a spherical surface that can be shown on the surfae of a plane, in which Meridians are cut by Parallels at right angles, and all very small shapes on the Sphere are represented by similar shapes on the plane.

51. In this most general solution is contained also that projection, with which one ordinarily the Hemispheres are mapped into the interior of circles, in

whose centers are placed the two poles.[10] This projection arises from the formulae established in §48:

$$x = s^{\alpha} \cos \alpha t, \qquad \text{and} \qquad y = -s^{\alpha} \sin \alpha t;$$

it is supposed that $\alpha = -1$, so that

$$x = \frac{\cos t}{\operatorname{tang}\left(45^\circ + \frac{1}{2}u\right)}, \qquad \text{and} \qquad y = \frac{\sin t}{\operatorname{tang}\left(45^\circ + \frac{1}{2}u\right)}.$$

For then, x and y vanish at the poles, where $u = 90^\circ$. But at the equator, where $u = 0$, and $s = 1$, then $x = \cos t$ and $y = \sin t$, whence

$$xx + yy = 1.$$

Thus, the Equator is represented by a circle whose radius=1. Moreover, at all points of longitude t,

$$\frac{y}{x} = \operatorname{tang} t,$$

so that every Meridian is represented by a radius of this circle. Finally, the image of the Parallel with latitude u is a circle, concentric with that of the Equator, and with radii

$$= \frac{1}{s} = \frac{1}{\operatorname{tang}\left(45^\circ + \frac{1}{2}u\right)} = \operatorname{tang}(45^\circ - \tfrac{1}{2}u);$$

that is, the radius is equal to the tangent of half the distance to the pole. And the hemispheres are customarily represented in conformity to this condition.

THIRD HYPOTHESIS

That all areas of the Earth are represented at their true size in the plane.

[10] Known today as the "polar stereographic projection"; the "Hemispheres" are the northern and southern hemispheres of the globe. Euler mentions this projetion again in [Eul1777b], §2, and [Eul1777c], §4.

52. We start with the general formulae for dx and dy (§8):

$$dx = p\,du + q\,dt, \quad \text{and} \quad dy = r\,du + s\,dt.$$

and suppose, once again, that all Meridians are cut by the Parallels at right angles. Thus, the condition $\frac{s}{q} = -\frac{p}{r}$ must be fulfilled. Accordingly, set $s = -np$, $q = +nr$, and we have

$$dx = p\,du + nr\,dt, \qquad dy = r\,du - np\,dt.$$

Now the element of a Meridian will be

$$PQ = du\,\sqrt{pp + rr},$$

and the element of a Parallel,

$$PR = n\,dt\,\sqrt{pp + rr}.$$

Therefore, the area of the rectangle $PQSR$ will be

$$n\,du\,dt\,(pp + rr)\,,$$

while, on the Sphere, the corresponding area $pqsr$ is

$$du\,dt\,\cos u.$$

Since both these two expressions must be equal, we obtain

$$n(pp + qq) = \cos u,$$

or

$$n = \frac{\cos u}{pp + rr},$$

so that on account of our Hypothesis we will have:

$$dx = p\,du + \frac{r\,dt\cos u}{pp + rr}, \qquad \text{and} \qquad dy = r\,du - \frac{p\,dt\cos u}{pp + rr}.$$

Therefore, it is required to find suitable functions for p and r, in order that these formulae be integrable.

53. To simplify the calculations, let us set

$$p = m\cos\phi \quad \text{and} \quad r = m\sin\phi,$$

so that $pp + rr = mm$, and we shall have

$$dx = m\,du\cos\phi + \frac{dt\cos u\sin\phi}{m},$$

and

$$dy = m\,du\sin\phi - \frac{dt\cos u\cos\phi}{m}.$$

Furthermore, set $m = k\cos u$, to obtain

$$dx = k\,du\cos u\cos\phi + \frac{dt\sin\phi}{k}$$

and

$$dy = k\,du\cos u\sin\phi - \frac{dt\cos\phi}{k}.$$

Finally, let us set $\cos u\,du = dv$, so that $v = \sin u$, resulting in

$$dx = k\,dv\cos\phi + \frac{dt\ \sin\phi}{k}, \qquad dy = k\,dv\sin\phi - \frac{dt\ \cos\phi}{k},$$

where now it is required to find suitable values for k and ϕ.

54. Since no suitable method is yet known for finding a general solution to these equations, we search for particular solutions. And first, the solution discovered above (§22), where $x = t$ and $y = \sin t$, immediately presents itself. These values arise from our fomulae, if in the latter we set $k = 1$ and $\phi = 90°$; and this extends to a more general solution if for k and ϕ are selected arbitrary constants. say $k = a$ and $\phi = \alpha$, whence is obtained

$$x = av\ \cos\alpha + \frac{t\sin\alpha}{a} \qquad \text{and} \qquad y = av\ \sin\alpha - \frac{t\cos\alpha}{a}.$$

This solution differs from the previous one only in that the the Meridians are no longer normal to our axis EF, but are inclined to it at an angle α. But the Parallels cut these Meridians at right angles and therefore are straight parallel lines.

55. We shall be able to elicit other solutions, if for one of the quantities k and ϕ we select a function which depends on v alone, and for the other a function which depends on ϕ alone. Then, if $k = T$ and $y = V$, we shall have

$$dx = T\,dv\,\cos V + \frac{dt}{T}\sin V$$

and

$$dy = T\,dv\,\sin V - \frac{dt}{T}\cos V.$$

From this it follows that

$$\begin{aligned} x &= T\int dv\,\cos V = \quad \sin V\int \frac{dt}{T}, \\ y &= T\int dv\,\sin V = -\cos V\int \frac{dt}{T}. \end{aligned}$$

The two expressions for x must be equal, as well as the two expressions for y.

56. From the equality of the two expressions for x we deduce that

$$\frac{\int dv\,\cos V}{\sin V} \quad = \quad \int \frac{dt}{T} : T \quad = \quad \alpha;$$

and from the equality of the two expressions for y, that

$$\frac{\int dv\,\sin V}{\cos V} \quad = \quad -\int \frac{dt}{T} : T \quad = \quad \beta.$$

From this, for the function T, the equalities

$$\int \frac{dt}{T} = \alpha T; \qquad \text{and} \qquad \int \frac{dt}{T} = -\beta T,$$

arise, and therefore it must be that $\beta = -\alpha$. Differentiating,

$$\frac{dt}{T} = \alpha\,dT \qquad \text{and so} \qquad T = \sqrt{\frac{2\,t}{\alpha}}.$$

For V on the other hand we have

$$\int dv\ \cos V = \alpha\ \sin V \qquad \text{and} \qquad \int dv\ \sin V = -\alpha\ \cos V;$$

Differentiating both of these gives $dv = \alpha\, dV$, so that $V = \dfrac{v}{\alpha}$, or, with a constant added,

$$V = \frac{v+c}{\alpha}.$$

57. From these values we find

$$\int \cos V\, dv = \alpha \sin V = \alpha \sin\left(\frac{v+c}{\alpha}\right) \quad \text{and} \quad \int \frac{dt}{T} = \alpha T = \sqrt{2\alpha t},$$

and the expressions for the two coordinates are found to be

$$x = \sin\left(\frac{v+c}{\alpha}\right)\sqrt{2\alpha t} \qquad \text{and} \qquad y = -\cos\left(\frac{v+c}{\alpha}\right)\sqrt{2\alpha t},$$

From this we immediately deduce

$$\sqrt{xx+yy} = \sqrt{2\alpha t},$$

and from this it is clear that all points with the same longitude t are located on the circumference of a circle with radius $\sqrt{2\alpha t}$. Therefore, in this representation, all Meridians are represented as concentric circles, and the prime Meridian, where $t = 0$, collapses into the common center point. The circles of parallel, therefore, are represented as radii of these circles. Clearly such a mapping is quite unsuitable, even if it fulfills all of the conditions which we are examining.

58. Now let k be a function of v alone, which $= V$, and the angle ϕ a function of t alone, which $= T$. We have

$$dx = V\,dv \cos T + \frac{dt\ \sin T}{V} \qquad \text{and} \qquad dy = V\,dv \sin T - \frac{dt\ \cos T}{V}$$

and from this the resulting values for x and y are:

$$x = \cos T \int V\,dv = \frac{1}{V}\int dt\ \sin T \qquad \text{and} \qquad y = \sin T \int V\,dv = -\frac{1}{V}\int dt\ \cos T.$$

From the equality of these expressions it is established that

$$V\int V\,dv = \frac{dt\,\sin T}{\cos T} = \alpha, \qquad \text{and} \qquad -V\int V\,dv = \frac{dt\,\cos T}{\sin T} = -\beta,$$

From the two expressions for V it follows immediately that $\alpha = \beta$;, then on differentiating,

$$V\,dv \;=\; -\frac{\alpha\,dV}{VV}, \qquad \text{or} \qquad dv \;=\; -\frac{\alpha\,dV}{V^3},$$

and integrating:

$$v + c = \frac{\alpha}{2VV}, \qquad \text{and from this,} \qquad V = \sqrt{\frac{\alpha}{2(v+c)}} \quad .$$

As for the function T,

$$\int dt\,\sin T = \alpha\cos T \qquad \text{and} \qquad -\int dt\,\cos T = \alpha\sin T;$$

differentiating these, it follows that

$$dT = -\frac{dt}{\alpha}, \qquad \text{and thus} \qquad T = -\frac{t}{\alpha}.$$

59. From the values discovered,

$$\int V\,dv = \sqrt{2\alpha(v+c)},$$

so that

$$x = \sqrt{2\alpha(v+c)}\,\cos\frac{t}{\alpha}, \qquad \text{and} \qquad y = -\sqrt{2\alpha(v+c)}\,\sin\frac{t}{\alpha},$$

from which it follows immediately that

$$\frac{y}{x} = -\operatorname{tang}\left(\frac{t}{\alpha}\right), \qquad \text{and} \qquad \sqrt{xx+yy} = \sqrt{2\alpha(v+c)}.$$

From the first of these formulae it is clear that all Meridians are represented as straight lines emerging as radii from a single fixed point. From the other formula it is clear that all Parallels are portrayed as concentric circles. Such a method of representation is very convenient for tracing a map of each hemisphere in the interior of a circle, whose center is the image of a pole. The shape of any region on the map does not differ appreciably from reality, and its true area can be measured directly from the map.[11],

60. In these three Hypotheses is contained everything ordinarily desired from geographic as well as hydrographic maps. The second Hypothesis treated above even covers all possible representations. But on account of the great generality of the resulting formulae, it is not easy to elicit from them any methods of practical use. Nor, indeed, was the intention of the present work to go into practical uses, especially since, with the usual projections, these matters have been explained in detail by others.

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[11]This projection was first presented by J.H. Lambert [Lam1772], and is commonly known today as "Lambert's azimuthal equal-area projection".

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