E447 Summatio Serierum ex Sin. et Cos. compositarum [Tom XVIII *Novi Comm. Acad. Sci. Petrop*. 1774, pp. 24-36]. This title consisted of two related papers, both of which are translated here. See also **E246**.

Summation of the series

$$\sin \phi^\lambda + \sin 2\phi^\lambda + \ldots \sin n\phi^\lambda$$

$$\cos \phi^\lambda + \cos 2\phi^\lambda + \ldots \cos n\phi^\lambda$$

§1. If[1] we put $p = \cos\phi + \sqrt{-1}\sin\phi$ and $q = \cos\phi - \sqrt{-1}\sin\phi$, it is known that

$$\cos n\phi = \frac{p^n + q^n}{2}, \quad \sin n\phi = \frac{p^n - q^n}{2\sqrt{-1}}$$

and also that $pq = 1$. Given these, it is evident that summations of such series can

[1] The λ power goes with the sin and cos, not with the ϕ. The λ's are positive integers. To keep as close as possible to Euler's notation without confusing the reader, I insert parentheses.

The expression $\int \sin nx^\alpha$ would be written nowadays as $\Sigma_{k=1}^n \sin^\alpha kx$. Euler's way of writing has some points in its favor, and I shall adhere to it, inserting parentheses where needed, thus: $\int (\sin nx)^\alpha$.

always be reduced to the two series or progressions

$$p^\alpha + p^{2\alpha} + ... + p^{n\alpha} = \tfrac{p^{(n+1)\alpha} - p^\alpha}{p^\alpha - 1} = p^\alpha \tfrac{1-p^{\alpha n}}{1-p^\alpha},$$
$$q^\alpha + q^{2\alpha} + ... + q^{n\alpha} = \tfrac{q^{(n+1)\alpha} - q^\alpha}{q^\alpha - 1} = q^\alpha \tfrac{1-q^{\alpha n}}{1-q^\alpha}.$$

§2. If now these two progressions are added to each other, producing Sum of the two series.

$$+p^\alpha + p^{2\alpha} + ... + p^{n\alpha}$$
$$+q^\alpha + q^{2\alpha} + ... + q^{n\alpha},$$

its sum will be

$$\frac{p^\alpha - p^{(n+1)\alpha}}{1-p^\alpha} + \frac{q^\alpha - q^{(n+1)\alpha}}{1-q^\alpha} =$$

$$= \frac{p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha + q^\alpha - q^{(n+1)\alpha} - p^\alpha q^\alpha + p^\alpha q^{(n+1)\alpha}}{1 - p^\alpha - q^\alpha + p^\alpha q^\alpha}$$

which transforms (since $pq = 1$) into

$$\frac{p^\alpha - p^{(n+1)\alpha} - 1 + p^{n\alpha} + q^\alpha - q^{(n+1)\alpha} - 1 + q^{n\alpha}}{2 - p^\alpha - q^\alpha}.$$

This reduces further, since

$$\begin{aligned} p^\alpha + q^\alpha &= 2\cos\alpha\phi, \\ p^{(n+1)\alpha} + q^{(n+1)\alpha} &= 2\cos(n+1)\alpha\phi, \\ p^{n\alpha} + q^{n\alpha} &= 2\cos n\alpha\phi, \end{aligned}$$

to this form:

$$\frac{\cos\alpha\phi - \cos(n+1)\alpha\phi - 1 + \cos n\alpha\phi}{1-\cos\alpha\phi} = -1 + \frac{\cos n\alpha\phi - \cos(n+1)\alpha\phi}{1-\cos\alpha\phi}$$

which is the sum of the proposed series.

§3. If however one of the series is subtracted from the other to give

Difference of the two series.

$$\begin{gathered} +p^\alpha + p^{2\alpha} + \ldots + p^{n\alpha} \\ -q^\alpha - q^{2\alpha} - \ldots - q^{n\alpha}, \end{gathered}$$

its sum will be

$$\frac{p^\alpha - p^{(n+1)\alpha}}{1-p^\alpha} + \frac{-q^\alpha + q^{(n+1)\alpha}}{1-q^\alpha}$$

which when brought to the same denominator becomes

$$\frac{+p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha - q^\alpha + q^{(n+1)\alpha} + p^\alpha q^\alpha - q^{(n+1)\alpha} p^\alpha}{1 - p^\alpha - q^\alpha + p^\alpha q^\alpha}.$$

This expression reduces on account of $pq = 1$ to

$$\frac{p^\alpha - q^\alpha - p^{(n+1)\alpha} + q^{(n+1)\alpha} - 1 + 1 + p^{n\alpha} - q^{n\alpha}}{2 - p^\alpha - q^\alpha}.$$

This reduces further, since

$$\begin{aligned}
p^{\alpha} - q^{\alpha} &= 2\sqrt{-1}\sin\alpha\phi, \\
p^{(n+1)\alpha} - q^{(n+1)\alpha} &= 2\sqrt{-1}\sin(n+1)\alpha\phi, \\
p^{n\alpha} - q^{n\alpha} &= 2\sqrt{-1}\sin n\alpha,
\end{aligned}$$

to this form:

$$\frac{\sin\alpha\phi - \sin(n+1)\alpha\phi + \sin n\alpha\phi}{1-\cos\alpha\phi}\sqrt{-1}.$$

§4. For the sake of brevity, let us denote the sums of these series by the last term, or the general term, preceded by the summation sign $\int$, so that the two cases we have worked out will show the summations $\int = \Sigma$

$$\begin{aligned}
\int(p^{n\alpha} + q^{n\alpha}) &= -1 + \tfrac{\cos n\alpha\phi - \cos(n+1)\alpha\phi}{1-\cos\alpha\phi}, \\
\int(p^{n\alpha} - q^{n\alpha}) &= \tfrac{\sin\alpha\phi - \sin(n+1)\alpha\phi + \sin n\alpha\phi}{1-\cos\alpha\phi}\cdot\sqrt{-1}.
\end{aligned}$$

These formulas will make it easy to solve all the cases considered below.

§5. Let first $\lambda = 1$, so that the two series to be summed are: $\lambda = 1.$

$$s = \sin\phi + \sin 2\phi + ... + \sin n\phi = \int\sin n\phi,$$

$$t = \cos\phi + \cos 2\phi + ... + \cos n\phi = \int\cos n\phi.$$

Since

$$\sin n\phi = \frac{p^n - q^n}{2\sqrt{-1}}, \quad \cos n\phi = \frac{p^n + q^n}{2},$$

we shall have

$$\begin{aligned} 2s\sqrt{-1} &= \int (p^n - q^n)\,, \\ 2t &= \int (p^n + q^n)\,. \end{aligned}$$

We immediately find from the preceding paragraph (with $\alpha = 1$)

$$\begin{aligned} 2s\sqrt{-1} &= \frac{\sin\phi + \sin n\phi - \sin(n+1)\phi}{1-\cos\phi}\sqrt{-1}, \\ 2t &= -1 + \frac{\cos n\phi - \cos(n+1)\phi}{1-\cos\phi} \end{aligned}$$

and so

$$\begin{aligned} s &= \frac{\sin\phi + \sin n\phi - \sin(n+1)\phi}{2(1-\cos\phi)}, \\ t &= -\frac{1}{2} + \frac{\cos n\phi - \cos(n+1)\phi}{2(1-\cos\phi)}. \end{aligned}$$

§6. Now let $\lambda = 2$, and again we set $\lambda = 2.$

$$\begin{aligned} s &= (\sin\phi)^2 + (\sin 2\phi)^2 + \cdots + (\sin n\phi)^2 = \int (\sin n\phi)^2\,, \\ t &= (\cos\phi)^2 + (\cos 2\phi)^2 + \cdots + (\cos n\phi)^2 = \int (\cos n\phi)^2\,. \end{aligned}$$

Since

$$\begin{aligned} (\sin n\phi)^2 &= \tfrac{p^{2n}-2p^nq^n+q^{2n}}{-4} &= \tfrac{1}{2} - \tfrac{p^{2n}+q^{2n}}{4}, \\ (\cos n\phi)^2 &= \tfrac{p^{2n}+2p^nq^n+q^{2n}}{4} &= \tfrac{1}{2} + \tfrac{p^{2n}+q^{2n}}{4}, \end{aligned}$$

we shall have these formulas : Typo.

$$\begin{aligned} 4s &= 2\int 1 - \int \left(p^{2n} + q^{2n}\right), \\ 4t &= 2\int 1 + \int \left(p^{2n} + q^{2n}\right). \end{aligned}$$

We have $\int 1 = n,$ since the number of terms is *n*; further (since $\alpha = 2$),

$$\int \left(p^{2n} + q^{2n}\right) = -1 + \frac{\cos 2n\phi - \cos 2(n+1)\phi}{1 - \cos 2\phi}.$$

If we substitute these values and divide by 4, we get

$$\begin{aligned} s &= \frac{n}{2} + \frac{1}{4} - \frac{\cos 2n\phi - \cos 2(n+1)\phi}{4(1 - \cos 2\phi)}, \\ t &= \frac{n}{2} - \frac{1}{4} + \frac{\cos 2n\phi - \cos 2(n+1)\phi}{4(1 - \cos 2\phi)}, \end{aligned}$$

whence immediately

$$s + t = n$$

as it clearly must be.

§7. Now we set $\lambda = 3$ and we represent the series to be summed as $\lambda = 3$

$$s = (\sin\phi)^3 + (\sin 2\phi)^3 + ... + (\sin n\phi)^3 = \int (\sin n\phi)^3,$$
$$t = (\cos\phi)^3 + (\cos 2\phi)^3 + ... + (\cos n\phi)^3 = \int (\cos n\phi)^3.$$

Since

$$\begin{aligned}(\sin n\phi)^3 &= \frac{p^{3n} - 3p^{2n}q^n + 3p^n q^{2n} - q^{3n}}{-8\sqrt{-1}},\\ (\cos n\phi)^3 &= \frac{p^{3n} + 3p^{2n}q^n + 3p^n q^{2n} + q^{3n}}{8},\end{aligned}$$

then, using $pq = 1$, we obtain: Typos

$$s = \frac{-1}{8\sqrt{-1}} \int \left(p^{3n} - q^{3n}\right) - \frac{3}{-8\sqrt{-1}} \int \left(p^n - q^n\right)$$

$$= \frac{-1}{8\sqrt{-1}} \int \left(p^{3n} - q^{3n}\right) + \frac{3}{8\sqrt{-1}} \int \left(p^n - q^n\right),$$

and

$$t = \frac{1}{8} \int \left(p^{3n} + q^{3n}\right) + \frac{3}{8} \int \left(p^n + q^n\right).$$

If we now substitute the values found above, the two sums will appear thus: Typos

$$\begin{aligned}s &= -\frac{\sin 3\phi - \sin 3(n+1)\phi + \sin 3n\phi}{8(1-\cos 3\phi)} + 3\frac{\sin\phi - \sin(n+1)\phi + \sin n\phi}{8(1-\cos\phi)},\\ t &= -\frac{1}{2} + \frac{\cos 3n\phi - \cos 3(n+1)\phi}{8(1-\cos 3\phi)} + 3\frac{\cos n\phi - \cos(n+1)\phi}{8(1-\cos\phi)}.\end{aligned}$$

§8. Now let $\lambda = 4$ so that we are seeking the sums $\lambda = 4$

$$s = (\sin\phi)^4 + (\sin 2\phi)^4 + ... + (\sin n\phi)^4 = \int (\sin n\phi)^4,$$

$$t = (\cos\phi)^4 + (\cos 2\phi)^4 + ... + (\cos n\phi)^4 = \int (\cos n\phi)^4.$$

Then we have

$$\begin{aligned}
(\sin n\phi)^4 &= \frac{p^{4n} - 4p^{3n}q^n + 6p^{2n}q^{2n} - 4p^n q^{3n} + q^{4n}}{16}, \\
(\cos n\phi)^4 &= \frac{p^{4n} + 4p^{3n}q^n + 6p^{2n}q^{2n} + 4p^n q^{3n} + q^{4n}}{16},
\end{aligned}$$

and on account of $pq = 1$ it follows that

$$\begin{aligned}
s &= \frac{1}{16}\int \left(p^{4n} + q^{4n}\right) - \frac{1}{4}\int \left(p^{2n} + q^{2n}\right) + \frac{3}{8}\int 1, \\
t &= \frac{1}{16}\int \left(p^{4n} + q^{4n}\right) + \frac{1}{4}\int \left(p^{2n} + q^{2n}\right) + \frac{3}{8}\int 1.
\end{aligned}$$

On substituting the values which we have given above, Typo

$$\begin{aligned}
s &= \frac{3}{8}n + \frac{3}{16} + \frac{\cos 4n\phi - \cos 4(n+1)\phi}{16(1-\cos 4\phi)} - \frac{\cos 2n\phi - \cos 2(n+1)\phi}{4(1-\cos 2\phi)}, \\
t &= \frac{3}{8}n - \frac{5}{16} + \frac{\cos 4n\phi - \cos 4(n+1)\phi}{16(1-\cos 4\phi)} + \frac{\cos 2n\phi - \cos 2(n+1)\phi}{4(1-\cos 2\phi)}.
\end{aligned}$$

In this fashion larger values of the exponent λ can be developed.

§9. If now we ask what sums of this type would become if the series continued infinitely far, several things would need to be looked into.[2] First, it is evident when λ is an even number that the sums of the series would be infinitely large if *n* were infinite. Yet if λ is an odd integer then nothing will make the sum infinitely large. The question comes down to assigning values for $\sin n\alpha\phi$ and $\cos n\alpha\phi$

Infinite series.

[2] See translator's comments at the end of this document.

when n is taken to be infinitely large. Those values could be anywhere between the outer limits[3] 1 and -1. If n were a finite number, then we could say nothing definite about the sum; if the final term approached one or the other of the outer limits, then the sum could be made to be anything. The illustrious author[4] of the preceding article chose values by ingenious reasoning that never-the-less was metaphysical; but we can settle on values perfectly well just using analysis.

§10. In these series as well as all other divergent series, the notion of the sum properly speaking is not adequate if we wish to determine a "sum" no matter what the final term may be. Wishing to stay with valid reasoning, I acknowledged to myself that a different sense of "sum" would be more appropriate for the analysis of these cases, and we should find as the "sum" of any infinite series, whether convergent or divergent, an analytic expression from which the series *arises*, and with this definition any doubts about its propriety would vanish.[5]

§11. To make this clearer, let us consider the first series developed above,

$$s = \sin\phi + \sin 2\phi + \sin 3\phi + \cdots + \sin n\phi$$

[3]The first "1" was left out.

[4]Dan. Bernoulli.

[5]"ut summa cuiusque seriei infinitae, sive fuerit convergens sive divergens, vocetur ea formula analytica, ex cuius evolutione eae series nascantur, hacque admissa definitione omnia dubia circa huiusmodi summationes sponte evanescunt."

which we found was

$$\frac{\sin\phi + \sin n\phi - \sin(n+1)\phi}{2(1-\cos\phi)}.$$

In this expression, the forms $\sin n\phi$ and $\sin(n+1)\phi$ enter on account of the final term. If the series is extended infinitely far, then with no final term these values eventually go away, and in this case the sum becomes

$$s = \frac{\sin\phi}{2(1-\cos\phi)}.$$

This is then the formula from which the series arises and so by my notion it can rightly be taken as the *sum* of that series. The same argument applies to the other series,

$$t = \cos\phi + \cos 2\phi + \cos 3\phi + \cdots + \cos n\phi$$

for which we found

$$t = -\frac{1}{2} + \frac{\cos n\phi - \cos(n+1)\phi}{2(1-\cos\phi)}.$$

If we omit the last fraction as depending only on the final term of the series, then by my notion the sum will be $t = -1/2$. As this may not be quite clear, this expression may be written as

$$t = \frac{\cos\phi - 1}{2(1-\cos\phi)},$$

which can be shown to be equivalent to the series: multiply each by $2 - 2\cos\phi$ and then we need

$$\begin{aligned}\cos\phi - 1 = &2\cos\phi + 2\cos 2\phi + 2\cos 3\phi + 2\cos 4\phi + \cdots \\ &- 2(\cos\phi)^2 - 2\cos\phi\cos 2\phi - 2\cos\phi\cos 3\phi - \cdots .\end{aligned}$$

Now, since

$$2\cos a\cos b = \cos(a-b) + \cos(a+b),$$

$$\begin{aligned}2(\cos\phi)^2 &= 1 + \cos 2\phi, & 2\cos\phi\cos 4\phi &= \cos 3\phi + \cos 5\phi,\\ 2\cos\phi\cos 2\phi &= \cos\phi + \cos 3\phi, & 2\cos\phi\cos 5\phi &= \cos 4\phi + \cos 6\phi,\\ 2\cos\phi\cos 3\phi &= \cos 2\phi + \cos 4\phi, & 2\cos\phi\cos 6\phi &= \cos 5\phi + \cos 7\phi,\end{aligned}$$

etc., and when we substitute in these values the equality is self-evident, for it becomes

$$\begin{array}{llllll}\cos\phi - 1 = & 2\cos\phi & +2\cos 2\phi & +2\cos 3\phi & +2\cos 4\phi & \\ & -1-\cos\phi & -\cos 2\phi & -\cos 3\phi & -\cos 4\phi & \text{etc..}\\ & & -\cos 2\phi & -\cos 3\phi & -\cos 4\phi & \end{array}$$

§12. With these observations in mind, we consider the case $\lambda = 3$. We had put $\lambda = 3$

$$\begin{aligned}s &= (\sin\phi)^3 + (\sin 2\phi)^3 + (\sin 3\phi)^3 + \cdots ,\\ t &= (\cos\phi)^3 + (\cos 2\phi)^3 + (\cos 3\phi)^3 + (\cos 4\phi)^3 + \cdots .\end{aligned}$$

The sums for these have the expressions

$$s = -\frac{\sin 3\phi}{8(1-\cos 3\phi)} + \frac{3\sin\phi}{8(1-\cos\phi)} \quad \text{and } t = -\frac{1}{2}.$$

To be sure, it is not immediately apparent how to derive the series from these expressions. The expert, though, will certainly see that these are correct. Still, it will help to show the truth of the latter summation. Since

$$(\cos a)^3 = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a,$$

the series resolves into two components, typo

$$\begin{aligned} t = \ & \tfrac{3}{4}\left(\cos\phi + \cos 2\phi + \cos 3\phi + \cos 4\phi + \cdots\right) \\ & + \tfrac{1}{4}\left(\cos 3\phi + \cos 6\phi + \cos 9\phi + \cdots\right). \end{aligned}$$

By the earlier formula, the sum of the first series is $\frac{3}{4}\cdot\left(-\frac{1}{2}\right) = -\frac{3}{8}$, and the sum of the second series is $\frac{1}{4}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{8}$ so that both joined together make the sum $-\frac{1}{2}$.

The General Summation of other Infinite Progressions Reducible to this Type

Theorem.

When the sum of the progression

$$Az + Bz^2 + Cz^3 + Dz^4 + \cdots + Nz^n$$

is known, then the progressions

$$S = Ax\sin\phi + Bx^2\sin 2\phi + Cx^3\sin 3\phi + \cdots + Nx^n\sin n\phi$$

and

$$T = Ax\cos\phi + Bx^2\cos 2\phi + Cx^3\cos 3\phi + \cdots + Nx^n\cos n\phi$$

can also be summed.

Demonstration.

The progression

$$Az + Bz^2 + Cz^3 + Dz^4 + \cdots + Nz^n$$

is a function of the variable quantity *z*; let it be denoted $\Delta : z$. As before, put $\Delta : z$

$$p = \cos\phi + \sqrt{-1}\sin\phi,$$
$$q = \cos\phi - \sqrt{-1}\sin\phi,$$

so that

$$\sin n\phi = \frac{1}{2\sqrt{-1}}\left(p^n - q^n\right)$$

and

$$\cos n\phi = \frac{1}{2}\left(p^n + q^n\right).$$

If these formulas are substituted into the proposed series, then we obtain for their sums the expressions

$$2S\sqrt{-1} = \Delta : px - \Delta : qx,$$
$$2T = \Delta : px + \Delta : qx.$$

It is to be noted that the imaginary parts of *p* and *q* will cancel out[6], so that *S* and *T*

[6] Actually, the real parts of the first equation and the imaginary parts of the second equation will cancel themselves.

will end up real. These formulas will hold equally well whether the series extend indefinitely or they end at some point.

Example 1.

Let all the coefficients $A = B = C = \ldots = 1$ with the series continued infinitely far. Then

$$\Delta : z = \frac{z}{1-z}.$$

For the first series

$$S = x \sin\phi + x^2 \sin 2\phi + x^3 \sin 3\phi + x^4 \sin 4\phi + ...$$

we get the expression

$$2S\sqrt{-1} = \frac{px}{1-px} - \frac{qx}{1-qx} = \frac{(p-q)x}{1-(p+q)x+pqx^2}.$$

Since $p - q = 2\sqrt{-1}\sin\phi$, $p + q = 2\cos\phi$ and $pq = 1$, this becomes

$$S = \frac{x\sin\phi}{1-2x\cos\phi+x^2}.$$

We get for the second series

$$T = x\cos\phi + x^2\cos 2\phi + x^3\cos 3\phi + \ldots$$

the expression

$$2T = \frac{px}{1-px} + \frac{qx}{1-qx} = \frac{(p+q)x - 2pqx^2}{1-(p+q)x+pqx^2},$$

or

$$T = \frac{x\cos\phi - x^2}{1 - 2x\cos\phi + x^2}.$$

Corollary 1.

For $x = 1$, these become the summations given above, namely

$$S = \frac{\sin\phi}{2(1-\cos\phi)} = \frac{1}{2}\cot\frac{1}{2}\phi$$

and

$$T = -\frac{1}{2}.$$

This latter is all the more worthy of notice, in that the individual terms of the series are variable, while the sum is constant.

Corollary 2.

There is always an *x*-value that can make the sum of the first series equal to any given quantity *a*. For this *x*-value, we have

$$\frac{x\sin\phi}{1-2x\cos\phi+x^2}=a.$$

If *x* is given a definite value, then the corresponding value of *a* can be expanded as

$$a=x\sin\phi+x^2\sin 2\phi+x^3\sin 3\phi+\cdots.$$

In a similar way, if we are given

$$\frac{x\cos\phi-x^2}{1-2x\cos\phi+x^2}=a,$$

then the value for *x* can be un-earthed, and we also have

$$a=x\cos\phi+x^2\cos 2\phi+x^3\cos 3\phi+\cdots.$$

Example 2.

Let now

$$\Delta : z=z+\frac{1}{2}z^2+\frac{1}{3}z^3+\frac{1}{4}z^4+\cdots=\log\frac{1}{1-z}.$$

Let then the proposed series be

$$S = x\sin\phi + \frac{1}{2}x^2\sin 2\phi + \frac{1}{3}x^3\sin 3\phi + \cdots$$

and

$$T = x\cos\phi + \frac{1}{2}x^2\cos 2\phi + \frac{1}{3}x^3\cos 3\phi + \cdots .$$

We shall have

$$2S\sqrt{-1} = \log\frac{1}{1-px} - \log\frac{1}{1-qx} = \log\frac{1-qx}{1-px},$$

or[7]

$$2S\sqrt{-1} = \log\frac{1 - x\cos\phi + x\sqrt{-1}\sin\phi}{1 - x\cos\phi - x\sqrt{-1}\sin\phi}.$$

To reduce this formula, consider the form

$$\log\frac{f + g\sqrt{-1}}{f - g\sqrt{-1}}.$$

If we put $g/f = \tan\omega$, then this logarithm will be $= 2\omega\sqrt{-1}$. The angle ω in our case must satisfy

$$\tan\omega = \frac{x\sin\phi}{1 - x\cos\phi}$$

and it follows immediately that *S* will equal this angle ω.

[7]log was missing.

For the other progression, from

$$2T = \log \frac{1}{1-px} + \log \frac{1}{1-qx} = -\log\left(1 - 2x\cos\phi + x^2\right)$$

comes

$$T = -\frac{1}{2}\log\left(1 - 2x\cos\phi + x^2\right).$$

Corollary.

From the first progression, that

$$\frac{x\sin\phi}{1 - x\cos\phi} = \tan\omega,$$

we derive

$$x = \frac{\tan\omega}{\sin\phi + \cos\phi\tan\omega} = \frac{\sin\omega}{\sin(\phi+\omega)}.$$

When we substitute this back[8], we find

$$\begin{aligned}\omega =& \frac{\sin\omega}{\sin(\phi+\omega)}\sin\phi + \frac{1}{2}\left(\frac{\sin\omega}{\sin(\phi+\omega)}\right)^2\sin 2\phi + \frac{1}{3}\left(\frac{\sin\omega}{\sin(\phi+\omega)}\right)^3\sin 3\phi \\ &+ \frac{1}{4}\left(\frac{\sin\omega}{\sin(\phi+\omega)}\right)^4\sin 4\phi + \cdots,\end{aligned}$$

which merits the greatest attention. Put $\omega = \pi/2$, so $\sin\omega = 1$ and $\sin(\phi+\omega) =$

[8] into the original series *S*.

$\cos\phi$. There follows the most beguiling summation

$$\frac{\pi}{2} = \frac{\sin\phi}{1}\left(\frac{1}{\cos\phi}\right) + \frac{\sin 2\phi}{2}\left(\frac{1}{\cos\phi}\right)^2 + \frac{\sin 3\phi}{3}\left(\frac{1}{\cos\phi}\right)^3 + \cdots .$$

Before, I had come upon this formula using various principles from differential calculus. It seemed all the more noteworthy there, that the series had the same constant sum no matter what we take for the angle ϕ.

Translator's Comments

Comments on "Summation of the Series ..."

The brevity of this article may mislead one into reading (and translating) it at speed, but there is much to ponder. At first, to be sure, Euler includes even simple steps, leading the reader to assume a didactic purpose to this article. And yet, in sections **6-8**, Euler did not bother to correct typos. His concern in these sections must be with the method of analysis rather than any particular answer.

In §**9**, Euler makes a serious jump in sophistication, addressing major issues. To prove that the sums do not become infinite when λ is an odd integer, he gives only the briefest of an argument that $\sin^\lambda n\phi$ and $\cos^\lambda n\phi$ will cover the interval [-1,1] with an even hand – perhaps he had in mind the Euler-MacLaurin summation formula with the summation over *n*, which clinches the matter in one step. Now, the argument does not work for certain values, such as $\phi = 0$ and $\phi = \pi/2$. So what did he mean by the general notion of a *limiting value*? Euler addresses first the question of what values $\sin n\alpha\phi$ and $\cos n\alpha\phi$ should take when *n* goes to infinity, leaving for §**10** the question of possible limiting values of the sums. Euler points out that no definite values can be attributed to the "final term", and excoriates the "illustrious author" of the article previous to his for resorting to metaphysics to pick some value for the final term, rather than relying solely on analysis, which is perfectly adequate to the main task of finding a value for the *sum*.

In §**1**0, Euler says that new notions and a new *definition* are required. Such notions, he says, could be used for both convergent *and divergent* cases. They would give results that would leave "no doubt of their correctness". He cannot do full justice to them in an article of nine pages, but he is on the path to Abel's theorem and Fourier analysis of step functions, and to a generalized notion of limit. He will content himself here with heuristic examples of the method, but his bold attack gives a significance to the article belied by its brevity.

In §**11**, Euler looks at the case $\lambda = 1$ and argues that the terms $\sin n\phi$ and $\sin(n+1)\phi$ in the formula

$$s_n = \frac{\sin\phi + \sin n\phi - \sin(n+1)\phi}{2(1-\cos\phi)}$$

need to dwindle in their effect if s_n is to reach some limit that is independent of *n*.

The limiting value must then be

$$\frac{\sin\phi}{2(1-\cos\phi)}.$$

This is the only candidate for an answer, and Euler is basically *defining* this answer as representing the sum of the series. In the same way, Euler argues that the only reasonable answer for the limit of the cosine series is -$\frac{1}{2}$. As it is not clear how a series can arise from this constant, Euler re-writes it as

$$t=\frac{\cos\phi-1}{2(1-\cos\phi)},$$

and then he shows that $2(1-\cos\phi)\cdot t$ will be $\cos\phi-1$ if you substitute for *t* its definition as a cosine series. These solutions for *s* and *t* are in fact the answers one would get by Abel's theorem.

Although he doesn't spell it out in detail, the notion that Euler has in mind is only one move away from Cesàro sums, and his limit is the limit determined by those sums. Cesàro sums suggest themselves from the formulas Euler derived for the initial values of λ – it would be second nature for him to see them as elements of a telescoping series. The important point here is that Euler recognized the need for a new *definition* rather than a new analytical *technique*. This is an awesome step.

It is worth noting that Euler calls his limit an *analytic expression* – that is, a function rather than a value. That is why the difficulties at $\phi=0$ and $\phi=\pi/2$ don't matter; they do not affect the possibility of a limiting function. In effect, Euler is giving a Fourier series and asking what function does it represent. His answer is that you *start* conceptually from the limiting function and verify that the given series is the appropriate Fourier representation of this function. And you can use any heuristic method to find exactly what that limiting function should be. Then, Euler feels, the question of convergence becomes moot: when the old conception of limit is inadequate for proving convergence, *you evolve your understanding of what a limit is*. A decent undergraduate course on Fourier analysis could be designed starting from this one paper.

Comments on "The General Summation..."

Euler plays his pet formula $e^{i\phi} = \cos\phi + i\sin\phi$ to the hilt. His quaint notation $\Delta : z$ for $\Delta(z)$ is too charming to resist.

Corollary 2 reverses the analysis of the first example to solve for example

$$\frac{x\sin\phi}{1 - 2x\cos\phi + x^2} = a,$$

giving *a* as a Fourier series with coefficients that are expressed in terms of *x*. This is the idea of a generating function. In the pyrotechnics of this corollary and **Example 2**, and especially in the final spectacular equation, we watch a star performer spin off his tricks so effortlessly.