Analytical Exercises *

Leonhard Euler

§1 Not little remarkable seems this relation, which I once observed to consist between the sums of these divergent series

$$1 - 2^m + 3^m - 4^m + 5^m - \text{etc.}$$

and these convergent ones

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

and which relation behaves as follows:

$$\begin{aligned}
1 - 2^0 + 3^0 - 4^0 + \text{etc.} &= \frac{1}{2}, \\
1 - 2^1 + 3^1 - 4^1 + \text{etc.} &= \frac{1}{4} = +\frac{2 \cdot 1}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}\right), \\
1 - 2^2 + 3^2 - 4^2 + \text{etc.} &= \frac{0}{8}, \\
1 - 2^3 + 3^3 - 4^3 + \text{etc.} &= -\frac{2}{16} = -\frac{2 \cdot 1 \cdot 2 \cdot 3}{\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}\right), \\
1 - 2^4 + 3^4 - 4^4 + \text{etc.} &= \frac{0}{32}, \\
1 - 2^5 + 3^5 - 4^5 + \text{etc.} &= \frac{16}{64} = +\frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi^6}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.}\right),
\end{aligned}$$

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$$1-2^6+3^6-4^6+\text{etc.} = \frac{0}{128},$$
$$1-2^7+3^7-4^7+\text{etc.} = -\frac{272}{256} = -\frac{2\cdot 1\cdot 2\cdot 3\cdots 7}{\pi^8}\left(1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.}\right),$$
$$1-2^8+3^8-4^8+\text{etc.} = \frac{0}{512},$$
$$1-2^9+3^9-4^9+\text{etc.} = \frac{7936}{1024} = +\frac{2\cdot 1\cdot 2\cdot 3\cdots 9}{\pi^{10}}\left(1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{etc.}\right),$$

where π denotes the circumference of the circle with diameter $=1$.

§2 From this it is possible to conclude, that in general there is a relation between these infinite series

$$1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.}$$

and these ones

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}$$

of this kind, that is

$$1-2^{n-1}+3^{n-1}-4^{n-1}+\text{etc.} = \frac{2\cdot 1\cdot 2\cdot 3\cdots (n-1)}{\pi^n}N\left(1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}\right),$$

where we indeed know, as often as n is an odd number, except in the case $n=1$, that N will be $=0$, but as often as n is an even number, N is either $+1$ or -1. Of course, if n is an oddly even number of the form $4m+2$, N will be $+1$, but if n is an evenly even number of the form $4m$, N will be -1. Hence it will possible without any difficulty to conclude, a function of what kind N is of n, because, if

$$n = \quad 2,\ 3,\quad 4,\ 5\quad 6,\ 7\quad 8,\ 9\quad 10,\ 11\quad 12,\ 13\quad \text{etc.},$$

it is

$$N = \quad +1,\ 0,\quad -1,\ 0\quad +1,\ 0\quad -1,\ 0\quad +1,\quad 0\quad -1,\quad 0\quad \text{etc.}$$

§3 And even, if we consider this with more attention, the case $n = 1$ does not violate this rule, according to which N has to become $= 0$; for nothing inhibits, that we allow this equality

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \frac{2}{\pi}\,0\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.}\right),$$

since the sum of the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

is infinite, whence it can certainly become

$$\frac{2}{\pi}\,0\,\infty = \frac{1}{2}$$

or equal to the sum of the series

$$1 - 1 + 1 - 1 + \text{etc.}$$

Therefore without any exception, if it was

$$n = 1, \qquad 2,\ 3, \qquad 4,\ 5 \qquad 6,\ 7 \qquad 8,\ 9 \quad \text{etc.},$$

it will be

$$N = 0, \quad +1,\ 0, \quad -1,\ 0 \quad +1,\ 0 \quad -1,\ 0 \quad \text{etc.}$$

which law can of course be satisfied by innumerous formulas to be assumed for N. But it cannot be doubted, that the simplest and most natural one is the right one here, which is

$$N = \cos\frac{n-2}{2}\pi,$$

while π denotes the angle equal to two right ones here, because the whole sine is assumed to be $= 1$, so that π is the half of the circumference of the circle.

§4 After having admitted this conjecture, we will therefore have in general

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.}$$
$$= 2\cos\frac{n-2}{2}\pi \cdot \frac{1\cdot 2\cdot 3\cdots(n-1)}{\pi^n}\left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \text{etc.}\right)$$

or by converting

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.}$$
$$= \frac{1}{2\cos\frac{n-2}{2}\pi} \cdot \frac{\pi^n}{1\cdot 2\cdot 3\cdots(n-1)}(1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.})$$

and from the preceding it is obvious, that this equality indeed holds, as often as n was an even number, and does not deviate from the truth in the cases, in which n is an odd number. Hence if it is true for the cases, in which n is a fractional number, the values of the formula $1\cdot 2\cdot 3\cdots(n-1)$ have to be assigned by interpolation, which indeed for the halves behave as follows: If $n-1 =$

$$\frac{1}{2}, \quad \frac{3}{2}, \quad \frac{5}{2}, \quad \frac{7}{2}, \quad \frac{9}{2} \quad \text{etc.}$$

they are

$$\frac{1}{2}\sqrt{\pi}, \quad \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi}, \quad \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi}, \quad \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2}\sqrt{\pi}, \quad \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 2\cdot 2\cdot 2\cdot 2}\sqrt{\pi} \quad \text{etc.}$$

and $\cos\frac{(n-2)\pi}{2} =$

$$\frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}}, \quad +\frac{1}{\sqrt{2}} \quad \text{etc.}$$

§5 Therefore we will have for these cases

$$1 + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \text{etc.}$$
$$= +\frac{\sqrt{2}}{1}\pi(1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} + \text{etc.})$$

$$1+\frac{1}{3^2\sqrt{3}}+\frac{1}{5^2\sqrt{5}}+\frac{1}{7^2\sqrt{7}}+\text{etc.}$$

$$=+\frac{2\sqrt{2}}{1\cdot 3}\pi^2(1-2\sqrt{2}+3\sqrt{3}-4\sqrt{4}+5\sqrt{5}+\text{etc.})$$

$$1+\frac{1}{3^3\sqrt{3}}+\frac{1}{5^3\sqrt{5}}+\frac{1}{7^3\sqrt{7}}+\text{etc.}$$

$$=-\frac{4\sqrt{2}}{1\cdot 3\cdot 5}\pi^3(1-2^2\sqrt{2}+3^2\sqrt{3}-4^2\sqrt{4}+5^2\sqrt{5}+\text{etc.})$$

$$1+\frac{1}{3^4\sqrt{3}}+\frac{1}{5^4\sqrt{5}}+\frac{1}{7^4\sqrt{7}}+\text{etc.}$$

$$=-\frac{8\sqrt{2}}{1\cdot 3\cdot 5\cdot 7}\pi^4(1-2^3\sqrt{2}+3^3\sqrt{3}-4^3\sqrt{4}+5^3\sqrt{5}+\text{etc.})$$

$$1+\frac{1}{3^5\sqrt{3}}+\frac{1}{5^5\sqrt{5}}+\frac{1}{7^5\sqrt{7}}+\text{etc.}$$

$$=+\frac{16\sqrt{2}}{1\cdot 3\cdot 5\cdot 7\cdot 9}\pi^5(1-2^4\sqrt{2}+3^4\sqrt{3}-4^4\sqrt{4}+5^4\sqrt{5}+\text{etc.}),$$

whether which equations are absolutely true, I have not dared to assure with absolute conviction; it is therefore convenient to scrutinise, whether it satisfies the series summed by approximation; and for the first we indeed calculate

$$1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\text{etc.}=0.380317$$

approximately, which number multiplied by $\pi\sqrt{2}$ gives 1.689655, which the sum of the series

$$1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\frac{1}{7\sqrt{7}}+\text{etc.}$$

is detected to be approximately equal to.

§6 But because it seems, that by taking odd numbers for n nothing can be concluded from that, since the one side of our equation tends to $\frac{0}{0}$, to investigate these values, let us take for n a number exceeding an integer infinitely less or let us write $n+\omega$ in the place of n, ω denoting an infinitesimal small fraction; and we will have

$$1+\frac{1}{3^{n+\omega}}+\frac{1}{5^{n+\omega}}+\frac{1}{7^{n+\omega}}+\text{etc.}$$

$$= \frac{1}{2\cos\frac{n-2+\omega}{2}\pi} \cdot \frac{\pi^{n+\omega}}{1\cdot 2\cdots(n-1+\omega)}(1 - 2^{n-1+\omega} + 3^{n-1+\omega} - 4^{n-1+\omega} + \text{etc.})$$

Here I therefore observe at first, that

$$\frac{1}{a^{n+\omega}} = a^{-n-\omega} = a^{-n}(1-\omega\ln a),$$

where the logarithms are to be understood to be natural or hyperbolic ones, so that

$$\frac{1}{a^{n+\omega}} = \frac{1}{a^n} - \frac{\omega\ln a}{a^n}.$$

In the same way it will be

$$a^{n-1+\omega} = a^{n-1} + a^{n-1}\omega\ln a$$

and

$$\pi^{n+\omega} = \pi^n(1+\omega\ln\pi);$$

then it indeed is

$$\cos\frac{n-2+\omega}{2}\pi = \cos\frac{n-2}{2}\pi - \frac{1}{2}\omega\pi\sin\frac{n-2}{2}\pi.$$

Since I finally once showed, that the value of the formula $1\cdot 2\cdots(n-1+\omega)$ in the case $n=1$ is $= 1 - 0.57721566\omega$, if we write for the sake of brevity

$$\lambda = 0.5572156649015328$$

by taking

$n=$	1,	2,	3,	4,	5	etc.

it is

$1\cdot 2\cdots(n-1+\omega) =$	$1-\lambda\omega$,	$1+(1-\lambda)\omega$,	$2+(3-2\lambda)\omega$,	$6+(11-6\lambda)\omega$,	$24+(50-24\lambda)\omega$	etc.

§7 Let us hence mainly consider the case $n = 3$, because this series

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.}$$

is conditioned in such a way, that all the labour up to now to investigate its sum, was without success. But because it is

$$\cos\frac{n-2}{2}\pi = 0 \quad \text{and} \quad \sin\frac{n-2}{2}\pi = 1,$$

our equation will take this form

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} - \omega\left(\ln 1 + \frac{\ln 3}{3^3} + \frac{\ln 5}{5^3} + \frac{\ln 7}{7^3} + \text{etc.}\right)$$

$$= \frac{-1}{\pi\omega}\cdot\frac{\pi^3(1+\omega\ln\pi)}{2+(3-2\lambda)\omega}\left(1 - 2^2 + 3^2 - 4^2 + \text{etc.} - \omega(2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - \text{etc.})\right).$$

But since

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0,$$

we will have for $\omega = 0$

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2}\pi^2(2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + \text{etc.})$$

and so we would achieve the aim, if it would be possible to assign the sum of this logarithmic series

$$2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + \text{etc.}$$

But in the same way one finds for the remaining powers

$$1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \text{etc.} = \frac{-\pi^4}{1\cdot 2\cdot 3\cdot 4}(2^4\ln 2 - 3^4\ln 3 + 4^4\ln 4 - 5^4\ln 5 + \text{etc.}),$$

$$1 + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \text{etc.} = \frac{+\pi^6}{1\cdot 2\cdots 6}(2^6\ln 2 - 3^6\ln 3 + 4^6\ln 4 - 5^6\ln 5 + \text{etc.}),$$

$$1 + \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \text{etc.} = \frac{-\pi^8}{1\cdot 2\cdots 8}(2^8\ln 2 - 3^8\ln 3 + 4^8\ln 4 - 5^8\ln 5 + \text{etc.})$$

etc.

§8 So let this infinite series be propounded to us

$$2^2\ln 2 - 3^2\ln 3 + 4^2\ln 4 - 5^2\ln 5 + 6^2\ln 6 - 7^2\ln 7 + \text{etc.} = Z,$$

that it is

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2}\pi\pi Z,$$

and to despair less about its sum, it should be noted, that it is

$$\ln 2 - \ln 3 + \ln 4 - \ln 5 + \ln 6 - \text{etc.} = \frac{1}{2}\ln\frac{\pi}{2}.$$

But that series Z can be transformed into several others, as for example

$$Z = \ln 2 - 3\ln\frac{3}{2} + 6\ln\frac{4}{3} - 10\ln\frac{5}{4} + 15\ln\frac{6}{5} - 21\ln\frac{7}{6} + \text{etc.}$$

and

$$\begin{aligned} Z = \ln\frac{2\cdot 2}{1\cdot 3} + 4\ln\frac{4\cdot 4}{3\cdot 5} + 9\ln\frac{6\cdot 6}{7\cdot 9} + 16\ln\frac{8\cdot 8}{7\cdot 9} + 25\ln\frac{10\cdot 10}{9\cdot 11} + \text{etc.} \\ -2\ln\frac{3\cdot 3}{2\cdot 4} - 6\ln\frac{5\cdot 5}{4\cdot 6} - 12\ln\frac{7\cdot 7}{6\cdot 8} - 20\ln\frac{9\cdot 9}{8\cdot 10} - \text{etc.} \end{aligned}$$

Since if we put in general

$$Z = \alpha\ln\frac{2\cdot 2}{1\cdot 3} - \beta\ln\frac{3\cdot 3}{2\cdot 4} + \gamma\ln\frac{4\cdot 4}{3\cdot 5} - \delta\ln\frac{5\cdot 5}{4\cdot 6} + \varepsilon\ln\frac{6\cdot 6}{5\cdot 7} - \zeta\ln\frac{7\cdot 7}{6\cdot 8} + \text{etc.},$$

it has to be

$$\begin{array}{lll} +2\alpha + \beta = 4 & \text{and hence} & \beta = 4 - 2\alpha, \\ \alpha + 2\beta + \gamma = 9 & & \gamma = 1 + 3\alpha, \\ \beta + 2\gamma + \delta = 16 & & \delta = 10 - 4\alpha, \\ \gamma + 2\delta + \varepsilon = 25 & & \varepsilon = 4 + 5\alpha, \\ \delta + 2\varepsilon + \zeta = 36 & & \zeta = 18 - 6\alpha, \\ \varepsilon + 2\zeta + \eta = 49 & & \eta = 9 + 7\alpha, \\ \zeta + 2\eta + \theta = 64 & & \theta = 28 - 8\alpha, \\ \eta + 2\theta + \iota = 81 & & \iota = 16 + 9\alpha, \\ \text{etc.} & & \text{etc.} \end{array}$$

Here we indeed took $\alpha = 1$, that the progression becomes as regular as possible.

§9 This last formula seems to be the most fitting for our purpose, because the logarithms are resolved into convergent series. For this aim I will use this resolution for the positive terms: Because any of them is contained in this form

$$xx \ln \frac{4xx}{4xx-1} = -xx \ln \left(1 - \frac{1}{4xx}\right),$$

from there this infinite series arises

$$xx\left(\frac{1}{4xx} + \frac{1}{2\cdot 2^4 x^4} + \frac{1}{3\cdot 2^6 x^6} + \frac{1}{4\cdot 2^8 x^8} + \text{etc.}\right)$$

or this one

$$\frac{1}{2^2} + \frac{1}{2\cdot 2^4}\cdot\frac{1}{xx} + \frac{1}{3\cdot 2^6}\cdot\frac{1}{x^4} + \frac{1}{4\cdot 2^8}\cdot\frac{1}{x^6} + \frac{1}{5\cdot 2^{10}}\cdot\frac{1}{x^8} + \text{etc.}$$

But for the ngative terms the general form is

$$-x(x+1)\ln\frac{(2x+1)^2}{4x(x+1)} = -x(x+1)\ln\left(1 + \frac{1}{4x(x+1)}\right),$$

which is resolved into this series

$$-\frac{1}{2^2} + \frac{1}{2\cdot 2^4}\cdot\frac{1}{x(x+1)} - \frac{1}{3\cdot 2^6}\cdot\frac{1}{x^2(x+1)^2} + \frac{1}{4\cdot 2^8}\frac{1}{x^3(x+1)^3} - \text{etc.},$$

whence the value of Z is transformed into these series

$$\begin{aligned}
Z = {} & \frac{1}{2^2}\left(1 - 1 + 1 - 1 + \text{etc.}\right) + \frac{1}{2\cdot 2^4}\left(1 + \frac{1}{1\cdot 2} + \frac{1}{2^2} + \frac{1}{2\cdot 3} + \frac{1}{3^2} + \text{etc.}\right)\\
& + \frac{1}{3\cdot 2^6}\left(1 - \frac{1}{1^2\cdot 2^2} + \frac{1}{2^4} - \frac{1}{2^2\cdot 3^2} + \text{etc.}\right) + \frac{1}{4\cdot 2^8}\left(1 + \frac{1}{1^3\cdot 2^3} + \frac{1}{2^6} + \frac{1}{2^3\cdot 3^3} + \frac{1}{3^6} + \text{etc.}\right)\\
& + \frac{1}{5\cdot 2^{10}}\left(1 - \frac{1}{1^4\cdot 2^4} + \frac{1}{2^8} - \frac{1}{2^4\cdot 3^4} + \text{etc.}\right) + \frac{1}{6\cdot 2^{12}}\left(1 + \frac{1}{1^5\cdot 2^5} + \frac{1}{2^{10}} + \frac{1}{2^5\cdot 3^5} + \frac{1}{3^{10}} + \text{etc.}\right)\\
& \text{etc.}
\end{aligned}$$

§10 If we hence for the sake of brevity set:

$$\begin{aligned}
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.} &= \alpha\pi^2,\\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.} &= \beta\pi^4,\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.} &= \gamma\pi^6,\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.} &= \delta\pi^8\\
&\text{etc.},
\end{aligned}$$

where the numbers α, β, γ, δ etc. are known, and because

$$1-1+1-1+\text{etc.}=\frac{1}{2},$$

it will be

$$\begin{aligned}
Z = \frac{1}{2^2}\cdot\frac{1}{2} &\qquad\qquad + \frac{1}{2\cdot 2^4}\left(\alpha\pi^2+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\text{etc.}\right)\\
&+\frac{1}{3\cdot 2^6}\left(\beta\pi^4-\frac{1}{2^2}-\frac{1}{6^2}-\frac{1}{12^2}-\text{etc.}\right)+\frac{1}{4\cdot 2^8}\left(\gamma\pi^4+\frac{1}{2^3}+\frac{1}{6^3}+\frac{1}{12^3}+\text{etc.}\right)\\
&+\frac{1}{3\cdot 2^{10}}\left(\delta\pi^8-\frac{1}{2^4}-\frac{1}{6^4}-\frac{1}{12^4}-\text{etc.}\right)+\frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10}+\frac{1}{2^5}+\frac{1}{6^5}+\frac{1}{12^5}+\text{etc.}\right)\\
&\text{etc.},
\end{aligned}$$

where the whole task is now already reduced to the summation of these series

$$\frac{1}{2^n}+\frac{1}{6^n}+\frac{1}{12^n}+\frac{1}{20^n}+\text{etc.},$$

the roots of whose powers - 2, 6, 12, 20 etc. - are the pronic numbers.

§11 But the single terms of this series, whose form is $\frac{1}{x^n(x+1)^n}$, can be resolved into parts of the simple powers, which parts behave as follows:

$$
\begin{aligned}
\frac{1}{x(x+1)} &= \frac{1}{x} - \frac{1}{x+1},\\
\frac{1}{x^2(x+1)^2} &= \frac{1}{x^2} + \frac{1}{(x+1)^2} - 2\left(\frac{1}{x} - \frac{1}{x+1}\right),\\
\frac{1}{x^3(x+1)^3} &= \frac{1}{x^2} - \frac{1}{(x+1)^3} - 3\left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right) + \frac{3\cdot 4}{1\cdot 2}\left(\frac{1}{x} - \frac{1}{x+1}\right),\\
\frac{1}{x^4(x+1)^4} &= \frac{1}{x^4} + \frac{1}{(x+1)^4} - 4\left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right) + \frac{4\cdot 5}{1\cdot 2}\left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right),\\
&\qquad - \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}\left(\frac{1}{x} - \frac{1}{x+1}\right)
\end{aligned}
$$

etc.

Because by indicating the sums with the prefixed sign $\int$ it is

$$\int \frac{1}{(x+1)^n} = \int \frac{1}{x^n} - 1,$$

it will be

$$
\begin{aligned}
\int \frac{1}{x(x+1)} &= 1,\\
\int \frac{1}{x^2(1+x)^2} &= 2\int \frac{1}{x^2} - 1 - 2,\\
\int \frac{1}{x^3(1+x)^3} &= 1 - 3\left(\int \frac{1}{x^2} - 1\right) + \frac{3\cdot 4}{1\cdot 2},\\
\int \frac{1}{x^4(1+x)^4} &= 2\int \frac{1}{x^4} - 1 - 4 + \frac{4\cdot 5}{1\cdot 2}\left(2\int \frac{1}{x^2} - 1\right) - \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}
\end{aligned}
$$

etc.

§12 In these single expressions it is possible to collect the absolute numbers to one in a convenient way, and because furthermore it is

$$\int \frac{1}{x^2} = \alpha\pi^2, \quad \int \frac{1}{x^4} = \beta\pi^4, \quad \int \frac{1}{x^6} = \gamma\pi^6, \quad \int \frac{1}{x^8} = \delta\pi^8 \quad \text{etc.},$$

we will have

$$\begin{aligned}
&\int \frac{1}{x(x+1)} &&= 1,\\
&\int \frac{1}{x^2(x+1)^2} &&= 2\alpha\pi^2 - \frac{2\cdot 3}{1\cdot 2},\\
&\int \frac{1}{x^3(1+x)^3} &&= -3\cdot 2\alpha\pi^2 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3},\\
&\int \frac{1}{x^4(1+x)^4} &&= 2\beta\pi^4 + \frac{4\cdot 5}{1\cdot 2}2\alpha\pi^2 - \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4},\\
&\int \frac{1}{x^4(1+x)^4} &&= -5\cdot 2\beta\pi^4 - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}2\alpha\pi^2 + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5},\\
&\int \frac{1}{x^6(1+x)^6} &&= 2\gamma\pi^6 + \frac{6\cdot 7}{1\cdot 2}2\beta\pi^4 + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}2\alpha\pi^2 - \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},\\
& &&\text{etc.},
\end{aligned}$$

where this remarkable reduction has to be observed

$$1+\frac{n}{1}+\frac{n(n+1)}{1\cdot 2}+\cdots+\frac{n(n+1)\cdots(2n-2)}{1\cdot 2\cdots(n-1)} = \frac{n(n+1)(n+2)\cdots(2n-1)}{1\cdot 2\cdot 3\cdots n};$$

for it is from a known law

$$= \frac{(n+1)(n+2)(n+3)\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(n-1)}.$$

§13 After having substituted these values we will obtain

$$Z = \frac{1}{2^2}\cdot\frac{1}{2} + \frac{1}{2\cdot 2^4}(\alpha\pi^2+1) + \frac{1}{3\cdot 2^6}\left(\beta\pi^4 + \frac{2\cdot 3}{1\cdot 2} - 2\alpha\pi^2\right)$$
$$+\frac{1}{4\cdot 2^8}\left(\gamma\pi^6 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3} - \frac{3}{1}\,2\alpha\pi^2\right) + \frac{1}{5\cdot 2^{10}}\left(\delta\pi^8 + \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4} - \frac{4\cdot 5}{1\cdot 2}\,2\alpha\pi^2 - 2\beta\pi^4\right)$$
$$+\frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\,2\alpha\pi^2 - \frac{5}{1}\,2\beta\pi^4\right)$$
$$+\frac{1}{7\cdot 2^{14}}\left(\zeta\pi^{12} + \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\,2\alpha\pi^2 - \frac{6\cdot 7}{1\cdot 2}\,2\beta\pi^4 - 2\gamma\pi^6\right) + \text{etc.},$$

which expression is resolved into these series

$$Z = \frac{1}{2^2\cdot 2} + \frac{1}{2\cdot 2^4} + \frac{2\cdot 3}{2\cdot 3\cdot 2^6} + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4\cdot 2^8} + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^{12}} + \text{etc.}$$

$$+\frac{\alpha\pi^2}{2\cdot 2^4} + \frac{\beta\pi^4}{3\cdot 2^6} + \frac{\gamma\pi^6}{4\cdot 2^8} + \frac{\delta\pi^8}{5\cdot 2^{10}} + \frac{\varepsilon\pi^{10}}{6\cdot 2^{12}} + \text{etc.}$$

$$\begin{aligned}
&-2\alpha\pi^2\left(\frac{1}{3\cdot 2^6} + \frac{3}{1\cdot 4\cdot 2^8} + \frac{4\cdot 5}{1\cdot 2\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 6\cdot 2^{12}} + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 7\cdot 2^{14}} + \text{etc.}\right)\\
&-2\beta\pi^4\left(\frac{1}{5\cdot 2^{10}} + \frac{5}{1\cdot 6\cdot 2^{12}} + \frac{6\cdot 7}{1\cdot 2\cdot 7\cdot 2^{14}} + \frac{7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 8\cdot 2^{16}} + \frac{8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 9\cdot 2^{18}} + \text{etc.}\right)\\
&-2\gamma\pi^6\left(\frac{1}{7\cdot 2^{14}} + \frac{7}{1\cdot 8\cdot 2^{16}} + \frac{8\cdot 9}{1\cdot 2\cdot 9\cdot 2^{18}} + \frac{9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 10\cdot 2^{20}} + \frac{10\cdot 11\cdot 12\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 11\cdot 2^{22}} + \text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}$$

§14 From this we are led to this general infinite series, that comprises all those numerical series:

$$\frac{1}{n\cdot 2^{2n}} + \frac{n}{(n+1)2^{2n+2}} + \frac{(n+1)(n+2)}{2(n+2)2^{2n+4}} + \frac{(n+2)(n+3)(n+4)}{2\cdot 3(n+3)2^{2n+6}}$$
$$+\frac{(n+3)(n+4)(n+5)(n+6)}{2\cdot 3\cdot 4(n+4)2^{2n+8}} + \text{etc.},$$

whose sum is therefore to be investigated. So if we indicate the sum of this series in general with this sign $S(n)$, we will have

$$Z = -\frac{1}{8} + S(1) + 2\alpha\pi^2\left(\frac{1}{4\cdot 2^4} - S(3)\right) + \beta\pi^4\left(\frac{1}{6\cdot 2^6} - S(5)\right)$$
$$+\gamma\pi^6\left(\frac{1}{8\cdot 2^8} - S(7)\right) + \delta\pi^8\left(\frac{1}{10\cdot 2^{10}} - S(9)\right) + \text{etc.}$$

But our general series can be exhibited more comfortable as follows:

$$n(n+1)S(n) = \frac{n+1}{2^{2n}} + \frac{nn}{2^{2n+2}} + \frac{n(n+1)(n+1)}{2\cdot 2^{2n+4}} + \frac{n(n+1)(n+2)(n+4)}{2\cdot 3\cdot 2^{2n+6}}$$
$$\frac{n(n+1)(n+3)(n+5)(n+6)}{2\cdot 3\cdot\cdot 2^{2n+8}} + \frac{n(n+1)(n+4)(n+6)(n+7)(n+8)}{2\cdot 3\cdot 4\cdot 5\cdot 2^{2n+10}} + \text{etc.},$$

where the denominators do not involve the number n. The single terms can also be represented by products in such a way, that one sets

$$S(n) = A + AB + ABC + ABCD + ABCDE + \text{etc.},$$

and it will be

$$A = \frac{1}{n\cdot 2^{2n}}, \quad B = \frac{nn}{4(n+1)}, \quad C = \frac{(n+1)(n+1)}{4\cdot 2n}, \quad D = \frac{(n+2)(n+4)}{4\cdot 3(n+1)},$$

$$E = \frac{(n+3)(n+5)(n+6)}{4\cdot 4(n+2)(n+4)}, \quad F = \frac{(n+4)((n+7)(n+8)}{4\cdot 5(n+3)(n+5)} \quad \text{etc.},$$

where one factor in general has this form

$$\frac{(n+\lambda-1)(n+2\lambda-3)(n+2\lambda-2)}{4\lambda(n+\lambda-2)(n+\lambda)}.$$

§15 Let us start with the simplest case $n = 1$, and because the factor is in general

$$= \frac{\lambda(2\lambda-2)(2\lambda-1)}{4\lambda(\lambda-1)(\lambda+1)} = \frac{2\lambda-1}{2\lambda+2},$$

it will be

$$A = \frac{1}{4}, \quad B = \frac{1}{8}, \quad C = \frac{3}{6}, \quad D = \frac{5}{8}, \quad E = \frac{7}{10}, \quad F = \frac{9}{12} \quad \text{etc.}$$

whence it is

$$S(1) = \frac{1}{4} + \frac{1}{4\cdot 8}\left(1 + \frac{3}{6} + \frac{3\cdot 5}{6\cdot 8} + \frac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10} + \frac{3\cdot 5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 12} + \text{etc.}\right).$$

But because it is

$$\sqrt{1-1} = 1 - \frac{1}{2} - \frac{1\cdot 1}{2\cdot 4} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10} - \text{etc.} = 0,$$

it will be

$$1 + \frac{3}{6} + \frac{3\cdot 5}{6\cdot 8} + \frac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10} + \text{etc.} = \frac{2\cdot 4}{1\cdot 1}\left(1 - \frac{1}{2}\right) = 4$$

and hence

$$S(1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

and

$$-\frac{1}{8} + S(1) = \frac{1}{4}.$$

§16 To to be able to determine the sums of the remaining series in an easier manner, let us write x in the place of $\frac{1}{2^2}$, that $x = \frac{1}{4}$, and because we have

$$S(n) = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{(n+1)(n+2)}{2(n+2)}x^{n+2} + \frac{(n+2)(n+3)(n+4)}{2 \cdot 3(n+3)}x^{n+3} + \text{etc.},$$

which in the case $n = 1$ passes over into

$$S(1) = x + \frac{1}{2}xx + \frac{2 \cdot 3}{2 \cdot 3}x^3 + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^4 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^5 + \text{etc.}$$

or

$$S(1) = x + \frac{1}{2}xx + x^3 + \frac{5}{2}x^4 + \frac{6 \cdot 7}{2 \cdot 3}x^5 + \frac{7 \cdot 8 \cdot 9}{2 \cdot 3 \cdot 4}x^6 + \frac{8 \cdot 9 \cdot 10 \cdot 11}{2 \cdot 3 \cdot 4 \cdot 5}x^7 + \text{etc.}$$

or

$$S(1) = x + \frac{1}{2}xx\left(1 + \frac{2}{1}x + \frac{2 \cdot 5}{1 \cdot 2}xx + \frac{2 \cdot 5 \cdot 14}{1 \cdot 2 \cdot 5}x^3 + \frac{2 \cdot 5 \cdot 14 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 5}x^4 + \text{etc.}\right)$$

or

$$S(1) = x + \frac{1}{2}xx\left(1 + \frac{3}{6}4x + \frac{3 \cdot 5}{6 \cdot 8}4^2xx + \frac{3 \cdot 5 \cdot 7}{6 \cdot 8 \cdot 10}4^3x^3 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 12}4^4x^4 + \text{etc.}\right);$$

but it is

$$\sqrt{1-4x} = 1 - \frac{1}{2}\,4x - \frac{1 \cdot 1}{2 \cdot 4}\,4^2x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\,4^3x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\,4^4x^4 - \text{etc.},$$

whence

$$\frac{1 \cdot 1}{2 \cdot 4}4^2x^2\left(1 + \frac{3}{6}\,4x + \frac{3 \cdot 5}{6 \cdot 8}4^2x^2 + \text{etc.}\right) = 1 - 2x - \sqrt{1-4x},$$

therefore

$$S(1) = x + \frac{1-2x-\sqrt{1-4x}}{4} = \frac{1+2x-\sqrt{1-4x}}{4},$$

and so it is for $x = \frac{1}{4}$

$$S(1) = \frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{3}{8},$$

as above.

§17 Now let us put $n = 3$ and let $S(3)$ be $= Q$, while

$$S(1) = P = \frac{1+2x-\sqrt{1-4x}}{4},$$

so that it is

$$\begin{aligned} P &= x + \frac{1}{2}xx \quad + \frac{2\cdot 3}{2\cdot 3}x^3 + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^4 + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^5 + \text{etc.},\\ Q &= \frac{1}{3}x^3 + \frac{3}{4}x^4 + \frac{4\cdot 5}{2\cdot 5}x^5 + \frac{5\cdot 6\cdot 7}{2\cdot 3\cdot 6}x^6 + \frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 7}x^7 + \text{etc.} \end{aligned}$$

From these one calculates

$$\begin{aligned} Pxx - Q = \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{2\cdot 3\cdot 5}{2\cdot 3\cdot 5}x^5 - \frac{3\cdot 4\cdot 5\cdot 8}{2\cdot 3\cdot 4\cdot 6}x^6 - \frac{4\cdot 5\cdot 6\cdot 7\cdot 11}{2\cdot 3\cdot 4\cdot 5\cdot 7}x^7 \\ - \frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 14}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8}x^8 - \text{etc.} \end{aligned}$$

and hence by differentiating

$$\begin{aligned} 2Px + \frac{xxdP}{dx} - \frac{dQ}{dx} = 2xx - x^3 - \frac{2\cdot 3}{2\cdot 3}\,5x^4 - \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}\,8x^5 - \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}\,11x^6 \\ - \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^8 - \text{etc.}, \end{aligned}$$

whose triple added to the first yields

$$2Px + \frac{4xxdP}{dx} - \frac{dQ}{dx} = 5xx + 2x^3 + 4 \cdot \frac{2 \cdot 3}{2 \cdot 3}x^4 + 4 \cdot \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^5 + 4 \cdot \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^6 + \text{etc.},$$

and

$$4Px = 4xx + 2x^3 + 4 \cdot \frac{2 \cdot 3}{2 \cdot 3}x^4 + 4 \cdot \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4}x^5 + 4 \cdot \frac{4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5}x^6 + \text{etc.},$$

hence

$$-2px + \frac{4xxdP}{dx} - \frac{dQ}{dx} = xx$$

and

$$dQ = 4xxdP - 2Pxdx - xxdx,$$

whence because of

$$dP = \frac{1}{2}dx + \frac{dx}{2\sqrt{1-4x}}$$

one calculates

$$dQ = -\frac{1}{2}xdx + \frac{1}{2}xdx\sqrt{1-4x} + \frac{2xxdx}{\sqrt{1-4x}} = -\frac{1}{2}xdx + \frac{xdx}{2\sqrt{1-4x}}$$

and by integrating

$$Q = -\frac{1}{4}xx - \frac{1+2x}{24}\sqrt{1-4x} + \frac{1}{24}.$$

Let x be $= \frac{1}{4}$, in which case $P = \frac{3}{8}$; Q will be $= \frac{5}{192}$, so that it is

$$S(1) = \frac{3}{8}, \quad S(3) = \frac{5}{192}$$

and

$$-\frac{1}{8} + S(1) = \frac{1}{4}, \quad \frac{1}{4 \cdot 2^4} - S(3) = -\frac{1}{96}.$$

§18 Now let us in general put $S(n) = P$ and the following sum $S(n+2) = Q$; it will be

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{n+1}{2}x^{n+2} + \frac{(n+2)(n+4)}{2\cdot 3}x^{n+3} + \frac{(n+3)(n+5)(n+6)}{2\cdot 3\cdot 4}x^{n+4} + \text{etc.},$$

$$Q = \frac{1}{n+2}x^{n+2} + \frac{n+2}{n+3}x^{n+3} + \frac{n+3}{2}x^{n+4} + \frac{(n+4)(n+6)}{2\cdot 3}x^{n+5} + \frac{(n+5)(n+7)(n+8)}{2\cdot 3\cdot 4}x^{n+6},$$

from which one concludes, that it will be

$$Q = Px - \frac{1}{2}(n+1)\int Pdx - \frac{1}{2}(n-1)xx\int \frac{Pdx}{xx}.$$

Hence from the value $S(n)$ one can define the value $S(n+2)$, except in the case $n = 1$, since then in $\int \frac{Pdx}{xx}$ there will appear $\int \frac{dx}{x}$.

But since the case $n = 3$ is already known

$$S(3) = \frac{1 - 6xx - (1+2x)\sqrt{1-4x}}{24},$$

if one takes this for P, it will be

$$S(5) = Px - 2\int Pdx - xx\int \frac{Pdx}{xx},$$

which expanded by integration gives

$$S(5) = \frac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{1-4x}).$$

Hence for $x = \frac{1}{4}$

$$S(5) = \frac{7}{32\cdot 60} = \frac{7}{1920}$$

and therefore

$$\frac{1}{6\cdot 2^6} - S(5) = -\frac{1}{960}.$$

§19 Let further n be $= 5$ and

$$P = \frac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{1-4x});$$

it will be

$$S(7) = Px - 3\int Pdx - 2xx \int \frac{Pdx}{xx},$$

after having expanded which integrals one finally finds

$$S(7) = \frac{1}{112}(1 - 28xx + 56x^3 - 14x^4 - (1 + 2x - 22xx + 20x^3)\sqrt{1-4x}).$$

For $x = \frac{1}{4}$ this therefore becomes

$$S(7) = \frac{1}{112}\left(1 - \frac{7}{4} + \frac{7}{8} - \frac{7}{128}\right) = \frac{9}{2^{11} \cdot 7}$$

and hence

$$\frac{1}{8 \cdot 2^8} - S(7) = -\frac{1}{2^{10} \cdot 7}.$$

So if we collect everything we found up to now, we will also easily obtain the following values by a conjecture:

$$\begin{aligned}
S(1) - \frac{1}{2 \cdot 2^2} &= \frac{1}{4} &&= \frac{1}{1 \cdot 2 \cdot 2^1}, \\
S(3) - \frac{1}{4 \cdot 4^2} &= \frac{1}{96} &&= \frac{1}{3 \cdot 4 \cdot 2^3}, \\
S(5) - \frac{1}{6 \cdot 2^6} &= \frac{1}{960} &&= \frac{1}{5 \cdot 6 \cdot 2^5}, \\
S(7) - \frac{1}{8 \cdot 2^8} &= \frac{1}{2^{10} \cdot 7} &&= \frac{1}{7 \cdot 8 \cdot 2^7}.
\end{aligned}$$

§20 Hence we finally obtain the following value for Z:

$$Z = \frac{1}{4} - \frac{\alpha\pi^2}{3 \cdot 4 \cdot 2^2} - \frac{\beta\pi^4}{5 \cdot 6 \cdot 2^4} - \frac{\gamma\pi^6}{7 \cdot 8 \cdot 2^6} - \frac{\delta\pi^8}{9 \cdot 10 \cdot 2^8} - \frac{\varepsilon\pi^{10}}{11 \cdot 12 \cdot 2^{10}} - \text{etc.},$$

so that

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\frac{1}{2}\pi\pi Z$$

or

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\frac{1}{8}\pi\pi-\frac{2\alpha\pi^2}{3\cdot 4\cdot 2^2}-\frac{2\beta\pi^4}{5\cdot 6\cdot 2^4}-\frac{2\gamma\pi^6}{7\cdot 8\cdot 2^6}-\text{etc.}$$

To investigate the sum of this series, let us consider π as a variable quantity and having set $\frac{\pi}{2}=\varphi$ let us put

$$\frac{\alpha\varphi^4}{3\cdot 4}+\frac{\beta\varphi^6}{5\cdot 6}+\frac{\gamma\varphi^8}{7\cdot 8}+\frac{\delta\varphi^{10}}{9\cdot 10}+\text{etc.}=s;$$

it will be

$$\frac{dds}{d\varphi^2}=\alpha\varphi^2+\beta\varphi^4+\gamma\varphi^6+\delta\varphi^8+\text{etc.}=z,$$

whence we form

$$\begin{aligned}2zz=2\alpha\alpha\varphi^4+4\alpha\beta\varphi^6&+4\alpha\gamma\varphi^8+4\alpha\delta\varphi^{10}+\text{etc.}\\ &+2\beta\beta\quad\;\; +4\beta\gamma.\end{aligned}$$

Now because

$$\beta=\frac{2\alpha\alpha}{5},\quad \gamma=\frac{4\alpha\beta}{7},\quad \delta=\frac{4\alpha\gamma+2\beta\beta}{9}\quad \text{etc.,}$$

it will be

$$2\int zzd\varphi=\beta\varphi^5+\gamma\varphi^7+\delta\varphi^9+\text{etc.}=z\varphi-\alpha\varphi^3$$

and hence

$$2zzd\varphi=zd\varphi+\varphi dz-3\alpha\varphi\varphi d\varphi.$$

§21 Because now $\alpha = \frac{1}{6}$, one finds by integration

$$z = \frac{1}{2} - \frac{\varphi}{2\tan\varphi},$$

as it will become clear to anyone trying. Hence, because

$$dds = zd\varphi^2,$$

one calculates

$$\frac{ds}{d\varphi} = \int zd\varphi = \frac{1}{2}\varphi - \frac{1}{2}\int\frac{\varphi d\varphi}{\tan\varphi}$$

and

$$s = \frac{1}{4}\varphi^2 - \frac{1}{2}\int d\varphi\int\frac{\varphi d\varphi}{\tan\varphi}$$

and hence

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{8}\pi\pi - \frac{1}{2}\varphi\varphi + \int d\varphi\int\frac{\varphi d\varphi}{\tan\varphi}$$

and because of $\varphi = \frac{\pi}{2}$ or $\pi = 2\varphi$ it will be

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \int d\varphi\int\frac{\varphi d\varphi}{\tan\varphi} = \frac{\pi}{2}\int\frac{\varphi d\varphi}{\tan\varphi} - \int\frac{\varphi\varphi d\varphi}{\tan\varphi}$$

$$= 2\int\varphi d\varphi\ln\sin\varphi - \frac{\pi}{2}\int d\varphi\ln\sin\varphi;$$

but it is

$$\int d\varphi\ln\sin\varphi = -\frac{\pi\ln 2}{2},$$

hence

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{\pi\pi}{4}\ln 2 + 2\int\varphi d\varphi\ln\sin\varphi,$$

where having taken the integrals in such a manner, that they vanish for $\varphi = 0$, thereafter one has to put $\varphi = \frac{\pi}{2}$, that one obtains the sum of the propounded series. But even if the integration cannot be done, its value can nevertheless be defined by quadratures. But the series found before by means of Z itself is very appropriate to determine the sum approximately.

§22 Hence I take the opportunity to consider this series more accurately

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{n+1}{2}x^{n+2} + \frac{(n+2)(n+4)}{2 \cdot 3}x^{n+3}$$
$$\frac{(n+3)(n+5)(n+6)}{2 \cdot 3 \cdot 4}x^{n+4} + \text{etc.},$$

whose values for the cases, in which n is an odd integer, we determined by means of a special method, which values behave as follows:

$$\begin{aligned}
\text{if}\quad n=1,\quad P &= \frac{1}{4}\left(1+2x - \sqrt{(1-4x)}\right),\\
n=3,\quad P &= \frac{1}{24}\left(1-6xx - (1+2x)\sqrt{1-4x}\right),\\
n=5,\quad P &= \frac{1}{60}\left(1-15xx+10x^3 - (1+2x-9xx)\sqrt{1-4x}\right),\\
n=7,\quad P &= \frac{1}{112}\left(1-28xx+56x^3-14x^4 - (1+2x-22xx+20x^3)\sqrt{1-4x}\right).
\end{aligned}$$

Therefore, because the summation in general can be reduced to an differential equation, it seems worth the effort to examine, how these values satisfy in this cases. But it will rather be convenient to consider the differentials, which are:

$$\begin{aligned}
\text{if}\quad n=1,\quad \frac{dP}{dx} &= \frac{1}{2}\left(1+\frac{1}{\sqrt{1-4x}}\right),\\
n=3,\quad \frac{dP}{dx} &= \frac{1}{2}\left(-x+\frac{x}{\sqrt{1-4x}}\right),\\
n=5,\quad \frac{dP}{dx} &= \frac{1}{2}\left(-x+xx+\frac{x-3xx}{\sqrt{1-4x}}\right),\\
n=7,\quad \frac{dP}{dx} &= \frac{1}{2}\left(-x+3xx-x^3\frac{x-5xx+5x^3}{\sqrt{1-4x}}\right).
\end{aligned}$$

§23 But in general by differentiating

$$\frac{dP}{dx} = x^{n-1} + nx^n + \frac{(n+1)(n+2)}{1 \cdot 2}x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3}x^{n+2} + \text{etc.}$$

Let us put

$$x = yy \quad \text{and} \quad \frac{dP}{dx} = \frac{dP}{2ydy} = s,$$

that we have

$$s = y^{2n-2} + ny^{2n} + \frac{(n+1)(n+2)}{1 \cdot 2}y^{2n+2} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3}y^{2n+4} + \text{etc.},$$

whence it becomes

$$y^{2-n}s = y^n + ny^{n+2} + \frac{(n+1)(n+2)}{1 \cdot 2}y^{n+4} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3}y^{n+6} + \text{etc.}$$

and hence further

$$\frac{dd(y^{2-n}s)}{dy^2} = n(n-1)y^{n-2} + \frac{n(n+1)(n+2)}{1}y^n + \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3}y^{2n+1} + \text{etc.},$$

which multiplied by y^{5-2n} and differentiated again produces

$$\frac{1}{2dy}d.\frac{s}{yy} = (n-2)y^{2n-5} + (n-1)y^{2n-3} + \frac{n(n+1)(n+2)}{1 \cdot 2}y^{2n-1} + \frac{(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2}y^5 + \text{etc.},$$

which series by the one above also is

$$= \frac{y^{3-n}dd(y^{2-n}s)}{dy^2},$$

so that we have this equation between s and y

$$d.\left(y^{5-2n}d.\frac{s}{yy}\right) = 4y^{3-n}dd(y^{2-n}s).$$

§24 For the element dy taken constant this equation expanded gives

$$\begin{aligned} & y^{3-2n}dds + (1-2n)y^{2-2n}dyds - 4(1-n)y^{1-2n}sdy^2 \\ = & 4y^{5-2n}dds + 8(2-n)y^{4-2n}dyds + 4(2-n)(1-n)y^{3-2n}sdy^2, \end{aligned}$$

which multiplied by y^{2n-1} becomes this one

$$\begin{aligned} & yy(1-4yy)dds + (1-2n)ydyds - 4(1-n)sdy^2 \\ & -8(2-n)y^3dyds - 4(2-n)(1-n)yysdy^2 = 0, \end{aligned}$$

which for $yy = x$ and for dx taken constant is transformed into this one

$$\begin{aligned} & xx(1-4x)dds + (1-n)xdxds - (1-n)sdx^2 \\ & -2(5-2n)xxdxds - (2-n)(1-n)sxdx^2 = 0, \end{aligned}$$

where it is

$$s = \frac{dP}{dx} \quad \text{and} \quad P = \int sdx.$$

But the integrations have to be executed according to this law, that for infinitely small x

$$\frac{ds}{dx} = (n-1)x^{n-2}, \quad s = x^{n-1} \quad \text{and} \quad P = \frac{1}{n}x^n.$$

§25 If this equation is integrated by means of an infinite series by beginning with the term x^{n-1}, the propounded series itself is reproduced, but the initial can also be the constant term x^0, whence also an integral is obtained, that does not satisfy our conditions; but furthermore one can choose another integral, that together with that one solves the task. Hence let us make the ansatz

$$s = O + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.},$$

it will be

$$\frac{ds}{dx} = A + 2Bx + 3Cxx + 4Dx^3 + 5Ex^4 + 6Fx^5 + \text{etc.}$$

and

$$\frac{dds}{dx^2} = 2B + 6Cx + 12Dxx + 20Ex^3 + 30Fx^4 + 42Gx^5 + \text{etc.},$$

after having substituted which series it has to become

$$\begin{array}{rrrrl}
 & & -(1-n)O & & \\
-(2-n)(1-n)Ox & -(2-n)(1-n)Ax^2 & -(2-n)(1-n)Bx^3 & -(2-n)(1-n)Cx^4 & -\text{etc.}=0, \\
+(1-n)A & +2(1-n)B & +3(1-n)C & +4(1-n)D & \\
-(1-n)A & -2(5-2n)A & -4(5-2n)B & -6(5-2n)C & \\
 & -(1-n)B & -(1-n)C & -(1-n)D & \\
 & +2B & +6C & +12D & \\
 & & -8B & -24C &
\end{array}$$

which equation is reduced to this form

$$\begin{array}{ccccc}
 & -(1-n)O & -(2-n)(1-n)O\ x & & \\
+(3-n)Bx^2 & +2(4-n)Cx^3 & +3(5-n)Dx^4 & +4(6-n)Ex^5 & +\text{etc.}=0. \\
+(3-n)(4-n)A & -(5-n)(6-n)B & -(7-n)(8-n)C & -(9-n)(10-n)D &
\end{array}$$

§26 Therefore having set the single terms equal to zero, O has to be $=0$, if n is not $=1$, but for the remaining coefficients one will have

$$\begin{aligned}
B &= \frac{(3-n)(4-n)}{1(3-n)}\ A = \frac{4-n}{1}A, \\
C &= \frac{(5-n)(6-n)}{2(4-n)}\ B = \frac{(5-n)(6-n)}{1\cdot 2}A, \\
D &= \frac{(7-n)(8-n)}{3(5-n)}\ C = \frac{(6-n)(7-n)(8-n)}{1\cdot 2\cdot 3}A, \\
E &= \frac{(9-n)(10-n)}{4(6-n)}D = \frac{(7-n)(8-n)(9-n)(10-n)}{1\cdot 2\cdot 3\cdot 4}A \\
&\qquad\qquad \text{etc.},
\end{aligned}$$

whence the law of the progression is obvious. But in the case $n=1$ the quantity O remains undetermined, but then the equation is satisfied after having set all remaining coefficients equal to zero, so that $s = O$, even though from these determinations also finite values could be assumed for them, as for example

$$B = \frac{3}{1}A, \quad C = \frac{4 \cdot 5}{1 \cdot 2}A, \quad D = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3}A \quad \text{etc.}$$

whence the complete integral will be

$$s = O + A\left(x + \frac{3}{1}x^2 + \frac{4 \cdot 5}{1 \cdot 2}x^3 + \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3}x^4 + \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}x^5 + \text{etc.}\right).$$

§27 In the same way for the remaining cases, in which n is a whole number, it is indeed $O = 0$, but the number A is arbitrary; but furthermore another certain coefficient is also not defined, which can therefore assumed as desired. Hence if it is put $= 0$, one will have an integral contained in finite terms, which will behave as follows:

$$\begin{array}{llllllll}
\text{if} & n = 3, & O = 0 & A \text{ is undefined,} & B = 0, & C = 0 & & \text{etc.;} \\
\text{if} & n = 4, & O = 0 & A \text{ is undefined,} & B = 0, & C = 0 & & \text{etc.;} \\
\text{if} & n = 5, & O = 0 & A \text{ is undefined,} & B = -A, & C = 0, & D = 0 & \text{etc.;} \\
\text{if} & n = 6, & O = 0 & A \text{ is undefined,} & B = -2A, & C = 0, & D = 0 & \text{etc.;} \\
\text{if} & n = 7, & O = 0 & A \text{ is undefined,} & B = -3A, & C = A, & D = 0 & \text{etc.;} \\
\text{if} & n = 8, & O = 0 & A \text{ is undefined,} & B = -4A, & C = \frac{2 \cdot 3}{1 \cdot 2}A, & D = 0 & \\
 & & & & & & E = 0 & \text{etc.;} \\
\text{if} & n = 9, & O = 0 & A \text{ is undefined,} & B = -5A, & C = \frac{3 \cdot 4}{1 \cdot 2}A & & \\
 & & & & D = -\frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3}A, & E = 0 \quad \text{etc.;} & &
\end{array}$$

if $n = 10$, $O = 0$ A is undefined, $B = -6A$, $C = \frac{4 \cdot 5}{1 \cdot 2} A$

$D = -\frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3} A$, $E = 0$ etc.;

if $n = 11$, $O = 0$ A is undefined, $B = -7A$, $C = \frac{5 \cdot 6}{1 \cdot 2} A$,

$D = -\frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} A$, $E = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} A$, $F = 0$ etc.;

etc.

§28 So look for all cases, in which n is a positive integer but not $n = 2$, at the particular integrals, whence the rational parts of the formulas found above for $\frac{dP}{dx}$ can be calculated:

$$
\begin{array}{lll}
\text{if} & n = 1, & s = O; \\
\text{if} & n = 3, & s = Ax; \\
\text{if} & n = 4, & s = Ax; \\
\text{if} & n = 5, & s = A(x - xx); \\
\text{if} & n = 6, & s = A(x - 2xx); \\
\text{if} & n = 7, & s = A(x - 3xx + x^3); \\
\text{if} & n = 8, & s = A(x - 4xx + 3x^3); \\
\text{if} & n = 9, & s = A(x - 5xx + 6x^3 - x^4); \\
\text{if} & n = 10, & s = A(x - 6xx + 10x^3 - 4x^4); \\
\text{if} & n = 11, & s = A(x - 7xx + 15x^3 - 10x^4 + x^5); \\
\text{if} & n = 12, & s = A(x - 8xx + 21x^3 - 20x^4 + 5x^6).
\end{array}
$$

§29 Hence for any number n this particular integral is

$$
s = A \left\{ \begin{array}{c} x - \frac{n-4}{1} xx + \frac{(n-5)(n-6)}{1 \cdot 2} x^3 - \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3} x^4 \\ + \frac{(n-7)(n-8)(n-9)(n-10)}{1 \cdot 2 \cdot 3 \cdot 4} x^5 - \text{etc.} \end{array} \right\},
$$

which series, even if continued to infinity, satisfies; nevertheless, although a certain term vanished, all following ones can be omitted, which taken alone would yield another particular integral. In addition it is evident from this, that any of these formulas is defined by the two preceding in such a way, that, if for the cases $n = \nu$, $n = \nu + 1$, $n = \nu + 2$ the values of s are put s, s', s'', it will be

$$s'' = s' - sx,$$

if the constant A retains the same value in all. And according to this law for the case $n = 2$ one has to put $s = 0$. But, as I already mentioned, these particular integrals do not satisfy our conditions, but give irrational parts, as we will see soon.

§30 But to find the complete integrals, let us investigate other particular integrals, which yield irrational parts. For this purpose let us put

$$s = \frac{t}{\sqrt{1-4x}};$$

it will be

$$ds = \frac{dt}{\sqrt{1-4x}} + \frac{2tdx}{(1-4x)^{\frac{3}{2}}}$$

and

$$dds = \frac{ddt}{\sqrt{1-4x}} + \frac{4dxdt}{(1-4x)^{\frac{3}{2}}} + \frac{12tdx^2}{(1-4x)^{\frac{5}{2}}},$$

having substituted which values our difference-differential equation will be converted into this form

$$\begin{aligned} xx(1-4x)ddt - (n-1)xdtdx + (n-1)tdx^2 \\ +2(2n-3)xxdtdx - n(n-1)txdx^2 = 0. \end{aligned}$$

So put here

$$t = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.}$$

and having done the substitution one gets to this equation

$$
\begin{array}{c}
0 = (n-1)A \\
-n(n-1)Ax - (n-3)Cx \qquad -2(n-4)Dx^3 \qquad -3(n-5)Ex^4 - \text{etc.}; \\
-(n-2)(n-3)B \quad -(n-4)(n-5)C \quad -(n-6)(n-7)D
\end{array}
$$

so if n is not $= 1$, A has to be $= 0$ and for the remaining

$$
\begin{aligned}
C &= -\frac{(n-2)(n-3)}{1(n-3)}B = -\frac{n-2}{1}B, \\
D &= -\frac{(n-4)(n-5)}{2(n-4)}C = -\frac{(n-2)(n-5)}{1\cdot 2}B, \\
E &= -\frac{(n-6)(n-7)}{3(n-5)}D = -\frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}B.
\end{aligned}
$$

§31 From these hence the finite values of t for the single integer numbers n will behave as follows:

$$
\begin{array}{lll}
\text{if} & n = 1, & t = A; \\
\text{if} & n = 2, & t = Bx; \\
\text{if} & n = 3, & t = Bx; \\
\text{if} & n = 4, & t = B(x - 2xx); \\
\text{if} & n = 5, & t = B(x - 3xx); \\
\text{if} & n = 6, & t = B(x - 4xx + 2x^3); \\
\text{if} & n = 7, & t = B(x - 5xx + 5x^3); \\
\text{if} & n = 8, & t = B(x - 6xx + 9x^3 - 2x^4); \\
\text{if} & n = 9, & t = B(x - 7xx + 14x^3 - 7x^4); \\
\text{if} & n = 10, & t = B(x - 8xx + 20x^3 - 16x^4 + 2x^5)
\end{array}
$$

and in general

$$
t = B\left\{
\begin{array}{c}
x - \dfrac{n-2}{1}xx + \dfrac{(n-2)(n-5)}{1\cdot 2}x^3 - \dfrac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}x^4 \\
+ \dfrac{(n-2)(n-7)(n-8)(n-9)}{1\cdot 2\cdot 3\cdot 4}x^5 - \text{etc.}
\end{array}
\right\},
$$

where, as above, again $t'' = t' - tx$.

§32 As we above denoted the series P by $S(n)$, let us denote the series $s = \frac{dP}{dx}$ by $\Sigma(n)$; it will be in general

$$\begin{aligned}
\Sigma(1) &= -A + \frac{B}{\sqrt{1-4x}}, \\
\Sigma(2) &= 0 + \frac{2Bx}{\sqrt{1-4x}}, \\
\Sigma(3) &= Ax + \frac{Bx}{\sqrt{1-4x}}, \\
\Sigma(4) &= Ax + \frac{B(x-2xx)}{\sqrt{1-4x}}, \\
\Sigma(5) &= A(x-xx) + \frac{B(x-3xx)}{\sqrt{1-4x}}, \\
\Sigma(6) &= A(x-2xx) + \frac{B(x-4xx+2x^3)}{\sqrt{1-4x}}, \\
\Sigma(7) &= A(x-3xx+x^3) + \frac{B(x-5xx+5x^3)}{\sqrt{1-4x}}, \\
\Sigma(8) &= A(x-4xx+3x^3) + \frac{B(x-6xx+9x^3-2x^4)}{\sqrt{1-4x}}, \\
\Sigma(9) &= A(x-5xx+6x^3-x^4) + \frac{B(x-7xx+14x^3-7x^4)}{\sqrt{1-4x}}, \\
\Sigma(10) &= A(x-6xx+10x^3-4x^4) + \frac{B(x-8xx+20x^3-16x^4+2x^5)}{\sqrt{1-4x}}
\end{aligned}$$

etc.;

here $A = -\frac{1}{2}$ and $B = \frac{1}{2}$, that these forms are accommodated to the propounded series.

§33 Because these values constitute a recurring series, because any of them becomes equal the last of the preceding less the penultimate multiplied by x, one will be able to express the general term or the value $\Sigma(n)$ in finite terms; then it will be from the property of recurring series

$$\Sigma(n) = M\left(\frac{1+\sqrt{1-4x}}{2}\right)^n + N\left(\frac{1-\sqrt{1-4x}}{2}\right)^n,$$

where from the first two coefficients M and N are determined in such a way, that

$$M = \frac{(A+B)(1-2x-\sqrt{1-4x})}{2x\sqrt{(1-4x}} = \frac{A+B}{x\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^n,$$

$$N = \frac{(B-A)(1-2x+\sqrt{1-4x})}{2x\sqrt{(1-4x}} = \frac{B-A}{x\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^n.$$

But because for our case $A = -\frac{1}{2}$ and $B = \frac{1}{2}$, it will be

$$\Sigma(n) = \frac{1}{x\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^2\left(\frac{1+\sqrt{1-4x}}{2}\right)^n$$

or

$$\Sigma(n) = \frac{x}{\sqrt{1-4x}}\left(\frac{1+\sqrt{1-4x}}{2}\right)^{n-2} = \frac{dP}{dx}$$

while

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{n+1}{2}x^{n+2} + \frac{(n+2)(n+4)}{2\cdot 3}x^{n+3} + \frac{(n+3)(n+5)(n+6)}{2\cdot 3\cdot 4}x^{n+4} + \text{etc.}$$

§34 Hence the value of this series, that we put P, is

$$P = \int \frac{xdx}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2},$$

to find which integral just put

$$\frac{1-\sqrt{1-4x}}{2} = y;$$

it will be

$$dy = \frac{dx}{\sqrt{1-4x}} \quad \text{and} \quad x = y - yy,$$

whence it is

$$P = \int dy(y - yy)y^{n-2} = \frac{y^n}{n} - \frac{y^{n+1}}{n+1}$$

and hence

$$P = \frac{n+1-ny}{n(n+1)} \cdot y^n$$

or

$$P = \frac{n+2+n\sqrt{1-4x}}{2n(n+1)} \left(\frac{1-\sqrt{1-4x}}{2} \right)^n = S(n)$$

for $x = \frac{1}{4}$, whence for the formulas exposed above (§ 14) one calculates

$$S(n) = \frac{n+2}{2^{n+1}n(n+1)}$$

and hence, as the formulas behave there,

$$\frac{1}{(n+1)2^{n+1}} - S(n) = -\frac{1}{2^n(n+1)n},$$

which expression agrees completely with those, we gave above (§ 19) based on induction alone, so that there indeed cannot further remain any doubt.

§35 Further it is remarkable, that the complete integral of this difference-differential equation

$$\begin{gathered} xx(1-4x)dds - (n-1)xdxds + (n-1)sdx^2 \\ +2(2n-5)xxdxds - (n-1)(n-2)sxdx^2 = 0 \end{gathered}$$

can be assigned and moreover algebraicly, which integral from the preceding behaves as follows:

$$s = \frac{Cx}{\sqrt{1-4x}} \left(\frac{1+\sqrt{1-4x}}{2} \right)^{n-2} + \frac{Dx}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2} \right)^{n-2};$$

how this can be found by integration from there, does not become clear so easily. Hence it is nevertheless immediately understood, that the substitution

$$s = \frac{xu}{\sqrt{1-4x}}$$

will be very helpful; then for $t = ux$ in § 30 this equation arises

$$xx(1-4x)ddu - (n-3)xdxdu - (n-2)(n-3)xudx^2$$
$$+2(2n-7)xxdxdu = 0$$

or

$$x(1-4x)ddu - (n-3)dxdu - (n-2)(n-3)udx^2$$
$$+2(2n-7)xdxdu = 0,$$

whose integral therefore is

$$u = C\left(\frac{1+\sqrt{1-4x}}{2}\right)^{n-2} + D\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2}.$$

§36 If one puts in this equation

$$\sqrt{1-4x} = y$$

and the element dy is introduced as a constant, this simpler equation will arise

$$(1-yy)ddu + 2(n-3)ydydu - (n-2)(n-3)udy^2 = 0,$$

whose integral is already known to be

$$u = C\left(\frac{1+y}{2}\right)^{n-2} + C\left(\frac{1-y}{2}\right)^{n-2}.$$

That it becomes clear, how this can be found from there, let us put $n = m+2$, that we have

$$(1-yy)ddu + 2(m-1)ydydu - m(m-1)udy^2 = 0,$$

where it is clear, that one convenient tries this proposition

$$u = (\alpha + \beta y)^m,$$

whence it becomes

$$du = m\beta dy(\alpha+\beta y)^{m-1} \quad \text{and} \quad ddu = m(m-1)\beta\beta dy^2(\alpha+\beta y)^{m-2},$$

whereafter it will be

$$m(m-1)(\alpha+\beta y)^{m-2}(\beta\beta(1-yy)+2\beta(\alpha y+\beta yy)-\alpha\alpha-2\alpha\beta y-\beta\beta yy)=0$$

and hence $\beta\beta=\alpha\alpha$, therefore

$$u=C(1\pm y)^m.$$

And because of the ambigous sign one will obtain the complete integral

$$u=C(1+y)^m+D(1-y)^m.$$

§37 Moreover it will be helpful to notice, that this last equation

$$(1-yy)ddu+2(m-1)ydydu-m(m-1)udy^2=0$$

becomes integrable, if it is divided by $(1\pm y)^m$. The first equation on the other hand

$$x(1-4x)ddu-(n-3)dxdu-(n-2)(n-3)udx^2+2(2n-7)xdxdu=0$$

will beocome integrable, if it is multiplied by

$$x^{-n+3}du-\frac{n-2}{2}x^{-n+2}udx.$$

But in general having propounded this equation

$$\begin{aligned}xx(A+Bx)ddu+\frac{1}{2}(2\alpha+\lambda)Axdxdu+\frac{1}{2}\alpha(\lambda-2)Audx^2\\+\frac{1}{2}(2\alpha+\lambda+1)Bxxdxdu+\frac{1}{2}\alpha(\lambda-1)Bxudx^2=0,\end{aligned}$$

if it is multiplied by

$$x^{\lambda-2}du+\alpha x^{\lambda-3}udx,$$

it will become integrable and the integral will be

$$\frac{1}{2}x^\lambda(A+Bx)du^2+\alpha x^{\lambda-1}(A+Bx)ududx+\frac{1}{2}\alpha\alpha x^{\lambda-2}(A+Bx)u^2dx^2=\frac{1}{2}Cdx^2$$

or

$$x^{\lambda}du^2 + 2\alpha x^{\lambda-1}ududx + \alpha\alpha x^{\lambda-2}u^2dx^2 = \frac{Cdx^2}{A+Bx},$$

therefore

$$x^{\frac{1}{2}\lambda}du + \alpha x^{\frac{1}{2}\lambda-1}udx = \frac{dx\sqrt{C}}{\sqrt{A+Bx}}$$

and hence

$$u = x^{-\alpha}\int\frac{x^{\alpha-\frac{1}{2}\lambda}dx\sqrt{C}}{\sqrt{A+Bx}}.$$

§38 But in this general equation ours from above is not contained; hence let us investigate conditions of this equation

$$xx(A+Bx)ddu + x(C+Dx)dudx + (E+Fx)udx^2 = 0$$

more accurately, that it multiplied by

$$x^{\lambda-2}du + \alpha x^{\lambda-3}udx$$

becomes integrable. And at first the integral is indeed

$$\frac{1}{2}x^{\lambda}(A+Bx)du^2 + \alpha x^{\lambda-1}(A+Bx)ududx + \frac{\alpha E}{\lambda-2}x^{\lambda-2}uudx^2$$
$$+\frac{\alpha F}{\lambda-1}x^{\lambda-1}u^2dx^2 = Gdx^2;$$

but it is required, that it is at first

$$C = \left(\alpha+\frac{1}{2}\lambda\right)A, \quad D = \left(\alpha+\frac{1}{2}\lambda+\frac{1}{2}\right)B,$$

but then indeed trifold

I. either $E = \frac{1}{2}\alpha(\lambda-2)A$ und $F = \frac{1}{2}\alpha(\lambda-1)B$, which is the superior case,

II. or $\lambda = 2\alpha+2$, $F = \frac{1}{2}\alpha(2\alpha+1)B$ while E remains undefined;

III. or $\lambda = 2\alpha+1$, $E = \frac{1}{2}\alpha(2\alpha-1)B$ while F remains undefined.

§39 So look at this two sufficiently far-reaching difference-differential equations, which can be integrated by this method:

$$\text{I.}\quad xx(A+Bx)ddu + (2\alpha+1)Axdxdu + Eudx^2$$
$$+\left(2\alpha+\frac{3}{2}\right)Bxxdxdu + \frac{1}{2}(2\alpha+1)Bxudx^2 = 0,$$

which multiplied by

$$x^{2\alpha}du + \alpha x^{2\alpha-1}udx$$

give the integral

$$\frac{1}{2}x^{2\alpha+2}(A+Bx)du^2 + \alpha x^{2\alpha+1}(A+Bx)ududx + \frac{1}{2}Ex^{2\alpha}uudx^2$$
$$+\frac{1}{2}\alpha\alpha Bx^{2\alpha+1}uudx^2 = Gdx^2.$$

The other form on the other hand is

$$\text{II.}\quad xx(A+Bx)ddu + \left(2\alpha+\frac{1}{2}\right)Axdxdu + \frac{1}{2}\alpha(2\alpha-1)Audx^2$$
$$+(2\alpha+1)Bxxdxdu + Fxudx^2 = 0,$$

which multiplied by

$$x^{2\alpha-1}du + \alpha x^{2\alpha-2}udx$$

yields this integral

$$\frac{1}{2}x^{2\alpha+1}(A+Bx)du^2 + \alpha x^{2\alpha}(A+Bx)ududx + \frac{1}{2}\alpha\alpha Ax^{2\alpha-1}u^2dx^2$$
$$+\frac{1}{2}Fx^{2\alpha}u^2dx^2 = Gdx^2.$$

If in the first one puts

$$A=1,\quad B=-4 \quad \text{und} \quad 2\alpha+1=-n+3 \quad \text{and} \quad E=0,$$

the equation propounded in § 35 arises.

§40 But there is another way to investigate the sum of the progression in § 23

$$\frac{dP}{dx} = x^{n-1} + \frac{n}{1}x^n + \frac{(n+1)(n+2)}{1\cdot 2}x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1\cdot 2\cdot 3}x^{n+2} + \text{etc.};$$

in this, because x is considered as a constant, let us consider this series

$$s = 1 + \frac{n}{2}a^2 + \frac{(n+1)(n+2)}{2\cdot 4}a^4 + \frac{(n+2)(n+3)(n+4)}{2\cdot 4\cdot 6}a^6 + \text{etc.},$$

where

$$aa = 2x \quad \text{and} \quad \frac{dP}{dx} = x^{n-1}s.$$

Now recall this series

$$\frac{(1+ay)^{-n+1} + (1-ay)^{-n+1}}{2} = 1 + \frac{(n-1)n}{1\cdot 2}aay^2 + \frac{(n-1)n(n+1)(n+2)}{1\cdot 2\cdot 3\cdot 4}a^4y^4 + \text{etc.},$$

for which for the sake of brevity we want to write

$$1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.},$$

and it will be

$$s = 1 + \frac{1}{n-1}Aa^2 + \frac{1\cdot 3}{(n-1)n}Ba^4 + \frac{1\cdot 3\cdot 5}{(n-1)(n+1)}Ca^6 + \text{etc.}$$

Now put

$$s = \frac{1}{z}\int dz(1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.})$$

and it has to be

$$\begin{aligned}
\int yydz &= \frac{1}{n-1}\int dz, \\
\int y^4dz &= \quad\frac{3}{n}\quad \int yydz, \\
\int y^6dz &= \frac{5}{n+1}\int y^4dz
\end{aligned}$$

and hence in general

$$\int y^{2\alpha} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz,$$

if after the integration a certain value is given to y.

§41 Hence let us put, that it is in general

$$\int y^{2\lambda} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz + \frac{Qy^{2\lambda - 1}}{n + \lambda - 2}$$

and by differentiating and dividing by $y^{2\lambda - 2}$ one calculates

$$(n - \lambda - 2) yydz = (2\lambda - 1) dz + ydQ + (2\lambda - 1) Qdy,$$

which equation has to hold for all numbers λ, whence it will be so

$$yydz = 2dz + 2Qdy$$

as

$$(n - 2) yydz = -dz + ydQ - Qdy,$$

therefore

$$dz = \frac{2Qdy}{yy - 2} = \frac{ydQ - Qdy}{(n - 2) yy + 1},$$

whence it becomes

$$\frac{dQ}{Q} = -\frac{(2n - 3) ydy}{2 - yy} \quad \text{and} \quad Q = (2 - yy)^{n - \frac{3}{2}}$$

and hence

$$dz = -2dy (2 - yy)^{n - \frac{1}{2}}.$$

Therefore having put $y = \sqrt{2}$ after the integration it is

$$\int y^{2\lambda} dz = \frac{2\lambda - 1}{n + \lambda - 2} \int y^{2\lambda - 2} dz$$

and one finds

$$s = \frac{\int dy(2-yy)^{n-\frac{1}{2}}((1+ay)^{-n+1}+(1-ay)^{-n+1}}{2\int dx(2-yy)^{n-\frac{1}{2}}},$$

if one puts $y=\sqrt{2}$ after the integration.

§42 Even though this method immediately exhibits the integral for the desired sum, it nevertheless does not show the true value in an algebraic expression. But above we saw, that

$$\frac{dP}{dx} = \frac{x}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2},$$

whence we colclude, that it will be here

$$s = \frac{x^{2-n}}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{n-2} = \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^{n-2}.$$

Hence if we put $2x = aa$, also the value of the superior integral formula in the case $y=\sqrt{2}$ will be algebraic

$$s = \frac{1}{\sqrt{1-2aa}}\left(\frac{1-\sqrt{1-2aa}}{aa}\right)^{n-2},$$

which circumstance by no means seems to be condemned, because it is maybe possible to derive many other beautiful results in this manner from there.