Synopsis by Section of Euler's

Remarks on a beautiful relation between direct as well as reciprocal power series (E 352)

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1. Euler states that he will find a beautiful relation involving the two series

(1) $$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + etc.$$

(2) $$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + etc,$$

where $n = m + 1$. While this relation is demonstrated only for special cases, its truth in general seems assured.

2. Euler explains his idea of the sum of a divergent series. Since the series 1-2+3-4+5-6 etc. arises by first expanding

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5^4 - 6x^5 + etc.,$$

then setting $x = 1$, we conclude that $1-2+3-4+5-6+etc. = 1/4$. In modern terms this is the Abel summation of the series.

3. Using calculus, Euler states (without explanation), that he can find the sums of series of the form $1 - 2^m x + 3^m x^2 - 4^m x^3 + etc.$, as rational functions of x. He lists these series with their sums for $m = 0, 1, 2, \cdots, 6$. A sample is

$$1-2^3x+3^3x^2-4^3x^3+etc.=\frac{1-4x+xx}{\left(1+x\right)^4}.$$

He then sets $x = 1$ and obtains the sums of series of the type (1) for $m=0,1,2,\cdots,9$.

A sample is

$$1-2^3+3^3-4^3+5^3-6^3+etc.=-\frac{2}{16}.$$

It is important to notice that when the power is $2m$, a positive even number, the sum is

$1-2^{2m}+3^{2m}-4^{2m}+5^{2m}-6^{2m}+etc.=0$.

4. Euler reviews his previous discovery of the sums

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+etc.=A(n)\pi^{2n},$$

for n a positive integer. The numbers $A(n)$ are rational and he has found their exact values. (Euler uses consecutive letters of the alphabet rather than $A(n)$.

$A(1)=A,\ A(2)=B,\ \cdots$.) Since

$$\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+etc.=\frac{2^{n-1}-1}{2^{n-1}}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+etc.\right),$$

he can now express the sum of series of the type (2) with powers that are even numbers. He gets

(3) $$\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\frac{1}{6^{2n}}+etc.=\frac{2^{2n-1}-1}{2^{2n-1}}A(n)\pi^{2n}.$$

He remarks that while he knows

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+etc.=\log 2,$$

his attempts to sum series of the type (2) for other odd powers have always failed.

5. Euler lists the exact numerical values of the important rational numbers $A(1)$ through $A(17)$.

6. He begins by stating that the series of the first type (1) also depend on the numbers $A(n)$. To show this he uses the Euler-Maclaurin summation formula, which he has found in a previous publication. He writes this formula in the unusual form

$$f(x)+f(x+\alpha)+f(x+2\alpha)+f(x+3\alpha)+f(x+4\alpha)+etc.=$$

$$-\frac{1}{\alpha}\int f(x)dx+\frac{1}{2}f(x)-\frac{\alpha A(1)df(x)}{2dx}+\frac{\alpha^3 A(2)d^3 f(x)}{2^3\,dx^3}-\frac{\alpha^5 A(3)d^5 f(x)}{2^5\,dx^5}+etc.$$

7. Using a simple algebraic manipulation, Euler converts the above result into a summation formula suitable for his alternating series

$$f(x)-f(x+\alpha)+f(x+2\alpha)-f(x+3\alpha)+f(x+4\alpha)-etc.$$

(4)

$$=\frac{1}{2}f(x)-\frac{\left(2^2-1\right)\alpha A(1)df(x)}{2dx}+\frac{\left(2^4-1\right)\alpha^3 A(2)d^3 f(x)}{2^3\,dx^3}-\frac{\left(2^6-1\right)\alpha^5 A(3)d^5 f(x)}{2^5\,dx^5}+etc..$$

To obtain series of the first type he sets $f(x)=x^m$ and $\alpha=1$ to get

$$x^m-(x+1)^m+(x+2)^m-(x+3)^m+(x+4)^m-(x+5)^m+etc.=$$

$$\frac{1}{2}x^m-\frac{m}{2}\left(2^2-1\right)A(1)x^{m-1}+\frac{m(m-1)(m-2)}{2\cdot\ \ 2\cdot\ \ \ 2}\left(2^4-1\right)A(2)x^{m-3}$$

$$-\frac{m(m-1)(m-2)(m-3)(m-4)}{2\cdot\ \ 2\cdot\ \ \ 2\cdot\ \ \ 2\cdot\ \ \ 2}\left(2^6-1\right)A(3)x^{m-5}+etc.,$$

where the sum on the right is finite if *m* is a non-negative integer.

8. Euler now lets $x=0$ in this last result to obtain series of the first type (1). On the right, all the terms disappear if *m* is even, and all but one term disappears if *m* is odd. He lists all the results for *m* = 0 to 10.

A sample is

$m=2$ $$1-2^2+3^2-4^2+5^2-6^2+etc.=0\,,$$

$m=3$ $$1-2^3+3^3-4^3+5^3-6^3+etc.=-1\cdot 2\cdot 3\frac{\left(2^4-1\right)}{2^3}A(2)\,.$$

These sums agree with the results of section 2, but only now is the dependence on $A(n)$ revealed.

9. When p is a positive integer, he has obtained the following two results:

$$1-2^{2p-1}+3^{2p-1}-4^{2p-1}+5^{2p-1}-6^{2p-1}+\cdots=(-1)^{p+1}(2p-1)!\frac{\left(2^{2p}-1\right)}{2^{2p-1}}A(p)\quad \text{, and}$$

$$\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\frac{1}{5^{2p}}-\frac{1}{6^{2p}}+\cdots=\frac{2^{2p-1}-1}{2^{2p-1}}A(p)\pi^{2p}\,.$$

Dividing the two he eliminates $A(p)$ and gets

$$\frac{1-2^{2p-1}+3^{2p-1}-4^{2p-1}+5^{2p-1}-6^{2p-1}+\cdots}{\frac{1}{1^{2p}}-\frac{1}{2^{2p}}+\frac{1}{3^{2p}}-\frac{1}{4^{2p}}+\frac{1}{5^{2p}}-\frac{1}{6^{2p}}+\cdots}=\frac{(-1)^{p+1}(2p-1)!\left(2^{2p}-1\right)}{\left(2^{2p-1}-1\right)\pi^{2p}}\,.$$

He also has

$$\frac{1-2^{2p}+3^{2p}-4^{2p}+5^{2p}-6^{2p}+\cdots}{\frac{1}{1^{2p+1}}-\frac{1}{2^{2p+1}}+\frac{1}{3^{2p+1}}-\frac{1}{4^{2p+1}}+\frac{1}{5^{2p+1}}-\frac{1}{6^{2p+1}}+\cdots}=0\,.$$

Euler lists this pair for p = 1 to 5.

.He ends by noting that the result that precedes these is (p = 0)

$$\frac{1-1+1-1+1-1+etc.}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+etc.}=\frac{1}{2\log 2}\,,$$

and does not, at first glance, seem connected to his discoveries.

10. Reflecting on the above results, Euler conjectures the general formula

(5) $$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+etc.}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+etc.}=\frac{-(n-1)!\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2},$$

which he has shown to be true for n = 2, 3, 4, … . This is the main result of the paper.

11. In this section Euler proves that his conjecture (5) is valid for n = 1. Since it is known that

$$\frac{1-1+1-1+1-1+etc.}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+etc.}=\frac{1}{2\log 2},$$

we must show that $\lim_{n\to 1}\frac{-(n-1)!\left(2^n-1\right)}{\left(2^{n-1}-1\right)\pi^n}\cos\frac{n\pi}{2}=\frac{1}{2\log 2}$. Euler does this using L'Hospital's rule.

12. Now Euler verifies his conjecture (5) for n = 0 which is known to be

$$\frac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+etc.}{1-1+1-1+1-1+etc.}=2\log 2.$$

As in the previous section, he uses L'Hospital's rule.

13. The conjecture (5) has been demonstrated for n = 0, 1, 2, 3, …. He now proves the conjecture for n a negative integer. He uses the identity $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin\pi x}$, which he has proved in a previous publication.

14. Euler verifies his conjecture for $n=1/2$ which is

$$\frac{1-\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}-\frac{1}{4^{1/2}}+\frac{1}{5^{1/2}}-\frac{1}{6^{1/2}}+etc.}{1-\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}-\frac{1}{4^{1/2}}+\frac{1}{5^{1/2}}-\frac{1}{6^{1/2}}+etc.}=\frac{-(-1/2)!\left(2^{1/2}-1\right)}{\left(2^{-1/2}-1\right)\pi^{1/2}}\cos\frac{\pi}{4}.$$

He uses $\Gamma(1/2)=\sqrt{\pi}$, and remarks that this case makes his conjecture very convincing.

15. Next Euler tests his conjecture for $n = 3/2$ which is

$$\frac{1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}+etc.}{1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}-\frac{1}{6\sqrt{6}}+etc.}=\frac{3+\sqrt{2}}{2\pi\sqrt{2}}=0.4967738\,.$$

He makes the verification numerically. To find the sum of the divergent series in the numerator he first calculates the sum of the first nine terms and gets

$$1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}+\sqrt{7}-\sqrt{8}+\sqrt{9}=1.9217396662\,.$$

From this he must subtract $\sqrt{10}-\sqrt{11}+\sqrt{12}-\sqrt{13}+\sqrt{14}-etc.$ which remarkably, he calculates using the alternating series version of the Euler-Maclaurin sum formula (4) from section 7. This is about 1.541610. Thus the numerator is 0.380129.

He calculates the sum of the convergent series in the denominator in the same way and gets 0.765158. The numerical agreement from the conjecture is remarkably close and Euler ends by saying “one could not doubt in the least that this matter is true.”

16. Convinced of the truth of his conjecture (5), Euler lists the special cases in which $n = 3/2, 5/2, 7/2, \ldots, 15/2$. A sample is

$$\frac{1-2^2\sqrt{2}+3^2\sqrt{3}-4^2\sqrt{4}+etc.}{1-\frac{1}{2^3\sqrt{2}}+\frac{1}{3^3\sqrt{3}}-\frac{1}{4^3\sqrt{4}}+etc.}=-\frac{1\cdot 3\cdot 5\left(8\sqrt{2}-1\right)}{2^3\left(8-\sqrt{2}\right)\pi^3}\,.$$

17. Euler continues his search to find the sum of the series

$$\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\frac{1}{7^n}-\frac{1}{8^n}+etc,$$

when n is an odd integer. He examines his conjecture (5) in this case:

$$1-\frac{1}{2^{2\lambda+1}}+\frac{1}{3^{2\lambda+1}}-\frac{1}{4^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-etc.=$$
$$-\frac{\left(2^{2\lambda}-1\right)\pi^{2\lambda+1}}{1\cdot 2\cdot 3\cdots 2\lambda\left(2^{2\lambda+1}-1\right)}\frac{\left(1-2^{2\lambda}+3^{2\lambda}-4^{2\lambda}+5^{2\lambda}-etc.\right)}{\cos\frac{2\lambda+1}{2}\pi}$$

and observes that the value of both the numerator

$$1-2^{2\lambda}+3^{2\lambda}-4^{2\lambda}+5^{2\lambda}-etc.$$

and the denominator $\cos\frac{2\lambda+1}{2}\pi=-\sin\lambda\pi$ are zero when λ is an integer. Thus we should try L'Hospital's rule.

18. Euler uses L'Hospital's rule and gets

$$1-\frac{1}{2^{2\lambda+1}}+\frac{1}{3^{2\lambda+1}}-\frac{1}{4^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-etc.=$$
$$\frac{2\left(2^{2\lambda}-1\right)\pi^{2\lambda}}{1\cdot 2\cdot 3\cdots 2\lambda\left(2^{2\lambda+1}-1\right)\cos\lambda\pi}(1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+etc.)\cdot \tag{6}$$

He lists several special cases and observes that summing the series

$$1^{2\lambda}\log 1-2^{2\lambda}\log 2+3^{2\lambda}\log 3-4^{2\lambda}\log 4+etc.$$

is probably more difficult than his original problem.

19. Since

$$1+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\frac{1}{9^m}+etc=\frac{2^m-1}{2(2^{m-1}-1)}\left(1-\frac{1}{2^m}+\frac{1}{3^m}-\frac{1}{4^m}+\frac{1}{5^m}-etc\right),$$

we get from the previous section

$$1+\frac{1}{3^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}+\frac{1}{7^{2\lambda+1}}+\frac{1}{9^{2\lambda+1}}+etc.=$$
$$-\frac{\pi^{2\lambda}}{1\cdot 2\cdot 3\cdots 2\lambda\cos\lambda\pi}(2^{2\lambda}\log 2-3^{2\lambda}\log 3+4^{2\lambda}\log 4-5^{2\lambda}\log 5+etc.). \tag{7}$$

Euler notes that (7) is slightly simpler than (6) and he lists a few special cases. Relation (7) is true only for $\lambda = 1, 2, 3, \cdots$. However, when $\lambda = 0$ we know that

$$\log 2 - \log 3 + \log 4 - \log 5 + etc. = \frac{1}{2}\log\frac{\pi}{2} \ ,$$

and this gives us hope for future research.

20. Euler states that he has found the similar conjecture

$$\frac{1 - 3^{n-1} + 5^{n-1} - 7^{n-1} + etc}{1 - 3^{-n} + 5^{-n} - 7^{-n} + etc} = \frac{1\cdot 2\cdot 3\cdots(n-1)2^n}{\pi^n}\sin\frac{n\pi}{2}$$

using the same methods. When $n = 2\lambda$, an even integer, we have:

$$1 - \frac{1}{3^{2\lambda}} + \frac{1}{5^{2\lambda}} - \frac{1}{7^{2\lambda}} + etc. = -\frac{\pi^{2\lambda-1}(3^{2\lambda-1}\log 3 - 5^{2\lambda-1}\log 5 + 7^{2\lambda-1}\log 7 - etc.)}{1\cdot 2\cdot 3\cdots\cdot(2\lambda-1)2^{2\lambda-1}\cos\lambda\pi},$$

which is similar to (7).

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